

## SINGULAR FOLIATIONS WITH TRIVIAL CANONICAL CLASS

FRANK LORAY<sup>1</sup>, JORGE VITÓRIO PEREIRA<sup>2</sup> AND FRÉDÉRIC TOUZET<sup>1</sup>

ABSTRACT. This paper is devoted to describe the structure of singular codimension one foliations with numerically trivial canonical bundle on projective manifolds. To achieve this goal we study the reduction modulo  $p$  of foliations, establish a criterium for uniruledness of projective manifolds, and investigate the deformation of free morphisms along foliations. This paper also contains new information about the structure of codimension one foliations on  $\mathbb{P}^n$  of degree smaller than or equal to  $2n - 3$ .

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## 1. INTRODUCTION AND STATEMENT OF RESULTS

Let  $\mathcal{F}$  be a singular holomorphic foliation on a compact complex manifold  $X$ , and  $K\mathcal{F}$  be its canonical line bundle. In strict analogy with the case of complex manifolds, the canonical line bundle of  $\mathcal{F}$  is the line bundle on  $X$  which, away from the singular set of  $\mathcal{F}$ , coincides with bundle of differential forms of maximal degree along the leaves of  $\mathcal{F}$ .

As in the case of manifolds, one expects that  $K\mathcal{F}$  governs much of the geometry of  $\mathcal{F}$ . When  $X$  is a projective surface, this vague expectation has already been turned into precise results. There is now a birational classification of foliations on projective surfaces, very much in the spirit of Enriques-Kodaira classification of projective surfaces, in terms of numerical properties of  $K\mathcal{F}$ , see [43, 12].

In this paper we investigate the structure of codimension one foliations on projective manifolds with  $K\mathcal{F}$  numerically equivalent to zero. We were dragged into this problem by a desire to better understand/generalize two previous results on the subject. The first, by Cerveau and Lins Neto [17], concerns the description of irreducible components of the space of foliations on  $\mathbb{P}^3$  with  $K\mathcal{F} = 0$ . While the

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second, by the third author of this paper [56], classifies smooth codimension one foliations with numerically trivial canonical bundle on compact Kähler manifolds.

Further motivation comes from the study of holomorphic Poisson manifolds as studied in [50], and also from [49, Corollary 4.6] which says that foliations with numerically trivial canonical bundle naturally appears when studying the obstructions for a projective variety to have  $\Omega_X^1$  generically ample.

**1.1. Previous results on foliations with  $c_1(K\mathcal{F}) = 0$ .** Cerveau and Lins Neto proved that the space of foliations on  $\mathbb{P}^3$  with  $K\mathcal{F} = 0$  has six irreducible components. Their result not only counts the number of irreducible components, but also give a rather precise description of them which we now proceed to recall. Four of the irreducible components parametrize foliations defined by logarithmic 1-forms with poles on a reduced divisor of degree four, the different irreducible components correspond to the different partitions of 4 with at least two summands. One of the components parametrizes pull-back under linear projections of foliations on  $\mathbb{P}^2$  with canonical bundle  $\mathcal{O}_{\mathbb{P}^2}(1)$ . The remaining component is rigid in the sense that its general element corresponds to a unique foliation up to automorphisms of  $\mathbb{P}^3$ . If we fix a point  $p$  in  $\mathbb{P}^1$  and identify  $\text{Aff}(\mathbb{C})$  with the isotropy group of this point under the natural action of  $\text{Aut}(\mathbb{P}^1)$  on  $\mathbb{P}^1$  then this foliation corresponds to the induced action of  $\text{Aff}(\mathbb{C})$  on  $\mathbb{P}^3 = \text{Sym}^3 \mathbb{P}^1$ . From this description one promptly sees that foliations on  $\mathbb{P}^3$  with trivial canonical bundle are either defined by closed rational 1-forms, or come from  $\mathbb{P}^2$  by means of a linear pull-back. Notice that in the latter case the leaves are covered by rational curves, indeed lines.

In 1997, one year after the publication of Cerveau-Lins Neto paper, appeared a paper [50] by Polishchuk which, among other things, contains a classification of Poisson structures on  $\mathbb{P}^3$  under restrictive hypothesis on their singular set. But (non-zero) Poisson structures on 3-folds are nothing more than foliations with trivial canonical bundle, thus Polishchuk's result is a particular case of Cerveau-Lins Neto classification.

The third author of this paper proved in [56] that a smooth codimension one foliation  $\mathcal{F}$  with numerically trivial canonical bundle on compact Kähler manifold  $X$  fits into at least one of the following descriptions.

- (1) After a finite étale covering,  $X$  is the product of Calabi-Yau variety  $Y$  and a complex torus  $T$  and  $\mathcal{F}$  is the pull-back under the natural projection to  $T$  of a linear codimension one foliation on  $T$ .
- (2) The manifold  $X$  is fibration by rational curves over a compact Kähler variety  $Y$  with  $c_1(Y) = 0$ , and  $\mathcal{F}$  is a foliation everywhere transverse to the fibers of the fibration.
- (3) The foliation  $\mathcal{F}$  is an isotrivial fibration by hypersurfaces with zero first Chern class.

The particular case of smooth Poisson structures on projective 3-folds was treated before by Druel in [26]. When the third author of this paper proved the classification above he was not aware of Druel's work.

**1.2. Rough structure of foliations with  $c_1(K\mathcal{F}) = 0$ .** While some of the foliations described above are defined by closed meromorphic 1-forms, some are not. Nevertheless, all of them are either defined by closed meromorphic 1-forms with coefficients in a torsion line bundle, or through a general point of the ambient

space there exists a rational curve contained in a leaf. Foliations having the latter property will be called uniruled foliations.

In face of these examples one is naturally lead to enquire if this pattern persists for arbitrary codimension one foliations with numerically trivial canonical bundle. One of our main results gives a positive answer to this question.

**Theorem 1.** *Let  $\mathcal{F}$  be a codimension one foliation with numerically trivial canonical bundle on a projective manifold  $X$ . Then at least one of following assertions holds true.*

- (a) *The foliation  $\mathcal{F}$  is defined by a closed rational 1-form with coefficients in a torsion line bundle and without divisorial components in its zero set.*
- (b) *All the leaves of  $\mathcal{F}$  are algebraic.*
- (c) *The foliation  $\mathcal{F}$  is uniruled.*

A codimension one foliation with all leaves algebraic is certainly defined by a closed rational 1-form, but this 1-form may have in general codimension one components in its zero set, thus (b) does not imply necessarily (a). In [19, Theorem 5.5], the reader will find explicit examples of foliations on  $\mathbb{P}^3$  that do not satisfy conclusions of Theorem 1; for these examples,  $K\mathcal{F} = \mathcal{O}_{\mathbb{P}^3}(k)$  with  $k \geq 3$ .

Our proof of Theorem 1 combines a variety of techniques: reduction modulo  $p$  of foliations, basic Hodge theory, deformations of morphisms along foliations, and the theory of transversely homogeneous structures for foliations. In the course of our investigations we stumble upon results with a somewhat broader scope, which we will now proceed to present.

**1.3. Semi-stable foliations and reduction modulo  $p$ .** In a joint work with Cerveau and Lins Neto, we have proved that codimension one foliations in positive characteristic are, as a rule, defined by closed rational 1-forms, see [19, Section 6]. Given a complex foliation  $\mathcal{F}$  on a projective manifold  $X$ , then both  $\mathcal{F}$  and  $X$  are defined over a finitely generated  $\mathbb{Z}$ -algebra  $R$  and we can reduce modulo a maximal prime  $\mathfrak{p} \subset R$  to obtain a foliation on a variety over a field of characteristic  $p > 0$ . Applying the above mentioned result we obtain that this reduction is indeed defined a closed rational 1-form. In general, one does not expect to be able to lift this information back to characteristic zero, since foliations on complex projective surfaces defined by closed 1-forms are quite rare. Nevertheless, under the additional assumption that  $T\mathcal{F}$  is semi-stable and  $K\mathcal{F}$  is numerically trivial we prove the following Theorem.

**Theorem 2.** *Let  $(X, H)$  be a  $n$ -dimensional polarized projective complex manifold, and  $\mathcal{F}$  be a semi-stable foliation of codimension one on  $X$ . If  $K\mathcal{F} \cdot H^{n-1} = 0$  then at least one of the following assertions holds true*

- (a) *for almost every maximal prime  $\mathfrak{p} \subset R$ , the reduction modulo  $\mathfrak{p}$  of  $\mathcal{F}$  is  $p$ -closed;*
- (b)  *$\mathcal{F}$  is induced by a closed rational 1-form with coefficients in a flat line bundle and without divisorial components in its zero set.*

In case (a) of Theorem 2, we expect that the foliation  $\mathcal{F}$  admits a rational first integral in characteristic zero. We are not alone in this hope. Ekedahl, Shepherd-Barron, and Taylor conjectured that this is the case for any foliation of any codimension [27]. Indeed their conjecture is a non-linear version of a previous conjecture by Grothendieck–Katz about the reduction modulo  $p$  of rational flat connections. To

the best of our knowledge, and despite of the recent advances [10], both conjectures are still wide-open up-to-date.

In a number of cases, we can deal with the  $p$ -closedness given by item (a) using some (basic) index theory for singularities of holomorphic foliations. For example, when there exists an ample divisor such that  $KX^2 \cdot H^{n-2} > 0$  we can prove that leaves of foliations as in (a) are covered by rationally connected varieties, see Section 3.4.

**1.4. A criterium for uniruledness.** Perhaps the first examples of foliations with  $K\mathcal{F} = 0$  that come to mind are those with trivial tangent bundle. Foliation with trivial tangent bundle are exactly those induced by (analytic) actions of complex Lie groups which are locally free outside an analytic subset of codimension at least two. If the action is not locally free then it is well-known that the manifold must be uniruled. We are able to generalize this well-known fact, confirming a recent conjecture of Peternell [49, Conjecture 4.23].

**Theorem 3.** *Let  $X$  be a projective manifold and  $L$  be a pseudo-effective line bundle on  $X$ . If there exists  $v \in H^0(X, \bigwedge^p TX \otimes L^*)$  not identically zero but vanishing at some point then  $X$  is uniruled. In particular, if there exists a foliation  $\mathcal{F}$  on  $X$  with  $c_1(T\mathcal{F})$  pseudo-effective and  $\text{sing}(\mathcal{F}) \neq \emptyset$  then  $X$  is uniruled.*

Theorem 3 brings the task of classifying codimension one foliations with  $c_1(K\mathcal{F}) = 0$  to the realm of uniruled manifolds, as smooth foliations satisfying these assumptions have already been classified by the third author. It is a consequence of Boucksom-Demailly-Păun-Peternell characterization of uniruledness [11] combined with the following result.

**Theorem 4.** *Let  $X$  be a projective manifold with  $KX$  pseudo-effective and  $L$  be a pseudo-effective line bundle on  $X$ . If  $v \in H^0(X, \bigwedge^p TX \otimes L^*)$  is a non-zero section then the zero set of  $v$  is empty. Moreover, if  $\mathcal{D}$  is a codimension  $q$  distribution on  $X$  with  $c_1(T\mathcal{D}) = 0$  then  $\mathcal{D}$  is a smooth foliation (i.e.  $T\mathcal{D}$  is involutive) with torsion canonical bundle, and there exists another smooth foliation  $\mathcal{G}$  of dimension  $q$  on  $X$  such that  $TX = T\mathcal{D} \oplus T\mathcal{G}$ .*

Theorem 4 also holds true in the Kähler realm, except for the claim that  $K\mathcal{D}$  is torsion as we use Simpson's Theorem [54], which is only available in the algebraic category, to prove it.

Using similar ideas we are able to prove that codimension one foliations with  $c_1(K\mathcal{F}) = 0$  having the so called division property are automatically smooth, see Theorem 4.6. As a corollary we prove that when  $F$  is not uniruled and has nonempty singular locus then  $\text{sing}(\mathcal{F})$  has an irreducible component of codimension two with non-vanishing Baum-Bott index, Corollary 4.9. This will be used in the proof of Theorem 1, and gives some evidence (rather weak we might say) toward Beauville's generalization of Bondal's conjecture on the degeneracy locus of holomorphic Poisson structures, see Remark 4.8.

**1.5. Foliation on uniruled manifolds.** On a uniruled variety we know there exist morphisms  $f : \mathbb{P}^1 \rightarrow X$  such that  $f^*TX$  is generated by global sections – the so called free morphisms. At a neighborhood of any free morphism  $f$  the irreducible component  $M = M_f$  of  $\text{Mor}(\mathbb{P}^1, X)$  containing  $f$  is smooth and has dimension  $h^0(\mathbb{P}^1, f^*TX)$ . A foliation  $\mathcal{F}$  on  $X$  naturally defines a foliation  $\mathcal{F}_{\text{tang}}$  on  $M_f$ . Intuitively, its leaves correspond to maximal families of morphisms which map

points on  $\mathbb{P}^1$  to leaves of  $\mathcal{F}$ . The dimension of  $\mathcal{F}_{tang}$  is equal to  $h^0(\mathbb{P}^1, f^*T\mathcal{F})$  where  $f$  is a general element of the irreducible component of  $\text{Mor}(\mathbb{P}^1, X)$  containing it. When  $c_1(K\mathcal{F}) = 0$ , we promptly see that  $h^0(\mathbb{P}^1, f^*T\mathcal{F}) \geq n - 1$ ,  $n = \dim(X)$ , and it is natural to expect that the study of  $\mathcal{F}_{tang}$  should shed light into the structure of  $\mathcal{F}$ . Indeed, this is true even if we do not assume  $K\mathcal{F} = 0$ . All we have to ask is the non-triviality of  $\mathcal{F}_{tang}$ , i.e.  $\dim \mathcal{F}_{tang} > 0$ , to be able to infer properties of the original foliation  $\mathcal{F}$ .

**Theorem 5.** *Let  $\mathcal{F}$  be a codimension one foliation on a  $n$ -dimensional uniruled projective manifold  $X$ . If  $f : \mathbb{P}^1 \rightarrow X$  is a general free morphism,  $\delta_0 = h^0(\mathbb{P}^1, f^*T\mathcal{F})$ , and  $\delta_{-1} = h^0(\mathbb{P}^1, f^*T\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^1}(-1))$  then at least one of the following assertions holds true.*

- (a) *The foliation  $\mathcal{F}$  is transversely projective.*
- (b) *The foliation  $\mathcal{F}$  is the pull-back by a rational map of a foliation  $\mathcal{G}$  on a projective manifold of dimension  $\leq n - \delta_0 + \delta_{-1}$ , and if  $\delta_{-1} > 0$  then  $\mathcal{F}$  is uniruled.*

*Moreover, if  $X$  is rationally connected and  $f$  is an embedding with ample normal bundle then we can replace transversely projective by transversely affine in item (a).*

To prove this result we use techniques germane to our previous joint works with Cerveau and Lins Neto [18, 19] combined with Bogomolov-McQuillan's graphic neighborhood [8]. In the case of rationally connected manifolds we add to the mixture a study of the variation of projective structures, see §6.6, together with Hartshorne's results on extension of meromorphic functions [31]. Using a Lefschetz-type Theorem due to Kollár [36], we derive from Theorem 5 the following consequence.

**Corollary 6.** *Let  $\mathcal{F}$  be a codimension one foliation on a rationally connected manifold  $X$ . If  $K\mathcal{F} = 0$  then  $\mathcal{F}$  is uniruled or defined by a closed rational 1-form.*

While it is still not our final word about the structure of foliations with  $c_1(K\mathcal{F}) = 0$ , the result above goes a long way in that direction, proving Theorem 1 when the ambient manifold is rationally connected.

Theorem 5 also admits as a Corollary a refinement of a recent result by Cerveau and Lins Neto [20] on the structure of foliations on  $\mathbb{P}^3$  with  $K\mathcal{F} = \mathcal{O}_{\mathbb{P}^3}(1)$  (foliations of degree three). They proved that a foliation of degree 3 on  $\mathbb{P}^3$  is either a pull-back of a foliation on  $\mathbb{P}^2$  by a rational map, or is transversely affine.

**Corollary 7.** *Let  $\mathcal{F}$  be a codimension one foliation on  $\mathbb{P}^n$  of degree  $d$ . If  $3 \leq d \leq 2n - 3$  then  $\mathcal{F}$  is a pull-back by a rational map of a foliation on a projective manifold of dimension at most  $\frac{d}{2} + 1$  or  $\mathcal{F}$  is defined by a closed rational 1-form.*

Our generalization is not only more precise, as we say defined by a closed rational 1-form while they say transversely affine, but also more general since it applies to projective spaces of every dimension greater than or equal to three.

**1.6. Ingredients of the proof of Theorem 1.** The results obtained from reduction modulo  $p$  (Theorem 2) together with the ones obtained from the study of deformations of free rational curves (Theorem 5) do not seem to imply directly Theorem 1. To deal with the cases not covered by them, we use Theorem 2 to restrain the type of singularities, and explore the existence of a projective structure given by Theorem 5 to infer the existence of an invariant divisor with good combinatorial

properties. To conclude we explore methods similar to the ones used in the proof of our criterium for uniruledness (Theorem 3).

**1.7. Plan of the paper.** In Section 2 we have collected basic results about foliations that will be used in the sequel. Section 3 is devoted to the reduction modulo  $p$  of foliations with  $K\mathcal{F} = 0$ . It starts by recalling results from [19] about the existence of invariant hypersurfaces and then proceeds to the proof of Theorem 2. This section finishes with the study of singularities of  $p$ -closed foliations and its implications to the structure of foliations with  $K\mathcal{F} = 0$ . Section 4 establishes our uniruledness criterium (Theorem 3). Section 5 reviews some of the theory of transversely projective foliations preparing the ground for Sections 6 and 7. Section 6 starts by recalling the basic theory of deformation of free morphisms from  $\mathbb{P}^1$  to projective manifolds, then it uses this theory to obtain naturally defined foliations on the space of such morphisms. The study of these foliations uncover some of the structure of the original foliation, and allow us to obtain Theorem 5. In Section 7 we put together information provided by Theorems 2, 3, and 5 and their proofs in order to establish our Theorem 1. In Section 8 we present a conjecture refining Theorem 1, together with some evidence toward it.

## 2. PRELIMINARIES

**2.1. Foliations as subsheaves of the tangent and cotangent bundles.** A foliation  $\mathcal{F}$  on a complex manifold is determined by a coherent subsheaf  $T\mathcal{F}$  of the tangent sheaf  $TX$  of  $X$  which

- (1) is closed under the Lie bracket (involutive), and
- (2) the inclusion  $T\mathcal{F} \rightarrow TX$  has torsion free cokernel.

The locus of points where  $TX/T\mathcal{F}$  is not locally free is called the singular locus of  $\mathcal{F}$ , denoted here by  $\text{sing}(\mathcal{F})$ .

Condition (1) allow us to apply Frobenius Theorem to ensure that for every point  $x$  in the complement of  $\text{sing}(\mathcal{F})$ , the germ of  $T\mathcal{F}$  at  $x$  can be identified with relative tangent bundle of a germ of smooth fibration  $f : (X, x) \rightarrow (\mathbb{C}^q, 0)$ . The integer  $q = q(\mathcal{F})$  is the codimension of  $\mathcal{F}$ . Condition (2) is of different nature and is imposed to avoid the existence of *removable* singularities. In particular it implies that the codimension of  $\text{sing}(\mathcal{F})$  is at least two.

The dual of  $T\mathcal{F}$  is the cotangent sheaf of  $\mathcal{F}$  and will be denoted by  $T^*\mathcal{F}$ . The determinant of  $T^*\mathcal{F}$ , i.e.  $(\wedge^p T^*\mathcal{F})^{**}$  where  $\dim(X) = n = p + q$ , in its turn is what we will call the canonical bundle of  $\mathcal{F}$  and will be denoted by  $K\mathcal{F}$ .

There is a dual point of view where  $\mathcal{F}$  is determined by a subsheaf  $N^*\mathcal{F}$  of the cotangent sheaf  $\Omega_X^1 = T^*X$  of  $X$ . The involutiveness asked for in condition (1) above is replaced by integrability: if  $d$  stands for the exterior derivative then  $dN^*\mathcal{F} \subset N^*\mathcal{F} \wedge \Omega_X^1$  at the level of local sections. Condition (2) is unchanged:  $\Omega_X^1/N^*\mathcal{F}$  is torsion free.

The normal bundle of  $\mathcal{F}$  is defined as the dual of  $N^*\mathcal{F}$ . Over the smooth locus  $X - \text{sing}(\mathcal{F})$  we have the following exact sequence

$$0 \rightarrow T\mathcal{F} \rightarrow TX \rightarrow N\mathcal{F} \rightarrow 0,$$

but this is no longer true over the singular locus. Anyway, as the singular set has codimension at least two we obtain the adjunction formula

$$KX = K\mathcal{F} \otimes \det N^*\mathcal{F}$$

valid in the Picard group of  $X$ .

The definitions above adapt verbatim to define algebraic foliations on smooth algebraic varieties defined over an arbitrary field. But be aware that the geometric interpretation given by Frobenius Theorem will no longer hold, especially over fields of positive characteristic.

**2.2. Harder-Narasimhan filtration.** Let  $\mathcal{E}$  be a torsion free coherent sheaf on a  $n$ -dimensional smooth projective variety  $X$  polarized by the ample line bundle  $H$ . The slope of  $\mathcal{E}$  (more precisely the  $H$ -slope of  $\mathcal{E}$ ) is defined as the quotient

$$\mu(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cdot H^{n-1}}{\text{rank}(\mathcal{E})}.$$

If the slope of every proper subsheaf  $\mathcal{E}'$  of  $\mathcal{E}$  satisfies  $\mu(\mathcal{E}') < \mu(\mathcal{E})$  (respectively  $\mu(\mathcal{E}') \leq \mu(\mathcal{E})$ ) then  $\mathcal{E}$  is called stable (respectively semi-stable). A sheaf which is semi-stable but not stable is said to be strictly semi-stable.

There exists a unique filtration of  $\mathcal{E}$  by torsion free subsheaves

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_r = \mathcal{E}$$

such that  $\mathcal{G}_i := \mathcal{E}_i / \mathcal{E}_{i-1}$  is semi-stable, and  $\mu(\mathcal{G}_1) > \mu(\mathcal{G}_2) > \cdots > \mu(\mathcal{G}_r)$ . This filtration is called the Harder-Narasimhan filtration of  $\mathcal{E}$ . Of course  $\mathcal{E}$  is semi-stable if and only if  $r = 1$ . Usually one writes  $\mu_{\max}(\mathcal{E}) = \mu(\mathcal{G}_1)$  and  $\mu_{\min}(\mathcal{E}) = \mu(\mathcal{G}_r)$ . The sheaf  $\mathcal{E}_1$  is called the maximal destabilizing subsheaf of  $\mathcal{E}$ . For the basic properties of torsion free coherent sheaves we are going to use see [34, Section 9.1.1] and references therein.

We will say that a foliation  $\mathcal{F}$  is stable/semi-stable/strictly semi-stable when its tangent sheaf  $T\mathcal{F}$  is stable/semi-stable/strictly semi-stable. The proof of [34, Chapter 9, Lemma 9.1.3.1] implies the following result.

**Proposition 2.1.** *Let  $\mathcal{F}$  be a foliation on a polarized smooth projective variety  $(X, H)$  satisfying  $\mu(T\mathcal{F}) \geq 0$ . If  $T\mathcal{F}$  is not semi-stable then the maximal destabilizing subsheaf of  $T\mathcal{F}$  is involutive. Thus there exists a semi-stable foliation  $\mathcal{G}$  tangent to  $\mathcal{F}$  and satisfying  $\mu(T\mathcal{G}) > \mu(T\mathcal{F})$ .*

*Proof.* Let  $\mathcal{E}$  be the maximal destabilizing subsheaf of  $T\mathcal{F}$ . Since  $T\mathcal{F}$  is involutive, the Lie bracket of local sections of  $\mathcal{E}$  lies in  $T\mathcal{F}$ . Thus the Lie bracket defines a morphism of  $\mathcal{O}_X$ -modules

$$[\cdot, \cdot] : \bigwedge^2 \mathcal{E} \longrightarrow \frac{T\mathcal{F}}{\mathcal{E}}.$$

On the one hand  $\mu(\bigwedge^2 \mathcal{E}) = 2\mu(\mathcal{E})$  and, since  $\mathcal{E}$  is semi-stable,  $\bigwedge^2 \mathcal{E}$  is semi-stable. On the other hand  $\mu_{\max}(T\mathcal{F}/\mathcal{E}) < \mu_{\max}(T\mathcal{F})$ . Therefore  $\mu(T\mathcal{F}) \geq 0$  implies  $\mu(\mathcal{E}) > 0$  and, consequently

$$\mu_{\min}(\bigwedge^2 \mathcal{E}) = \mu(\bigwedge^2 \mathcal{E}) > \mu(\mathcal{E}) = \mu_{\max}(T\mathcal{F}) > \mu_{\max}(T\mathcal{F}/\mathcal{E}).$$

But  $\mu_{\min}(\mathcal{A}) > \mu_{\max}(\mathcal{B})$  implies  $\text{Hom}_{\mathcal{O}_X}(\mathcal{A}, \mathcal{B}) = 0$  for any pair of torsion free sheaves. We conclude that  $\mathcal{E}$  is involutive and must be equal to the tangent sheaf of a foliation  $\mathcal{G}$ .  $\square$

**2.3. Miyaoka-Bogomolov-McQuillan Theorem.** The result stated below is a particular case of a more general result by Bogomolov and McQuillan proved in [8]. It generalizes a Theorem of Miyaoka, see [45, Theorem 8.5], [34, Chapter 9], or [33].

**Theorem 2.2.** *Let  $\mathcal{F}$  be foliation on a complex projective manifold  $X$ . If there exists a curve  $C \subset X$  disjoint from the singular set of  $\mathcal{F}$  for which  $T\mathcal{F}|_C$  is ample then the leaves of  $\mathcal{F}$  intersecting  $C$  are algebraic and the closure of a leaf of  $\mathcal{F}$  through a general point of  $C$  is a rationally connected variety.*

We recall that a variety  $Y$  is rationally connected if through any two points  $x, y \in Y$  there exists a rational curve  $C$  in  $Y$  containing  $x$  and  $y$ . Foliations with all leaves algebraic and with rationally connected general leaf will be called rationally connected foliations. Notice that there are rationally connected foliation with some leaves non rationally connected. For example, if we consider the codimension one foliation on  $\mathbb{P}^3$  determined by a pencil of cubics generated by  $3H$ , an hyperplane with multiplicity three, and a cone  $V$  over a smooth planar cubic transverse to  $H$  then the general leaf is a smooth cubic surface, and therefore rationally connected, but  $V$  is not rationally connected but only rationally chain-connected. Thus this foliation is rationally connected but it has one leaf which is not.

We will use Theorem 2.2 in the following form, closer to Miyaoka's original statement.

**Corollary 2.3.** *Let  $\mathcal{F}$  be a semi-stable foliation on a  $n$ -dimensional polarized projective variety  $(X, H)$ . If  $K\mathcal{F} \cdot H^{n-1} < 0$  then  $\mathcal{F}$  is a rationally connected foliation.*

*Proof.* If  $m \gg 0$  and  $C$  is a very general curve defined as a complete intersection of elements of  $|mH|$  then  $T\mathcal{F}|_C$  is a semi-stable vector bundle of positive degree according to [44, Theorem 6.1]. Therefore every quotient bundle of  $T\mathcal{F}|_C$  has positive degree and we can apply [30, Theorem 2.4] to see that  $T\mathcal{F}|_C$  is ample. We apply Theorem 2.2 to conclude.  $\square$

**2.4. Tangent subvarieties and pull-backs.** Let  $\mathcal{F}$  be a singular foliation on a projective manifold  $X$  of dimension  $n$ . We will say that  $\mathcal{F}$  is the pull-back of a foliation  $\mathcal{G}$  defined on a lower dimensional variety  $Y$ , say of dimension  $k < n$ , if there exists a dominant rational map  $\pi : X \dashrightarrow Y$  such that  $\mathcal{F} = \pi^*\mathcal{G}$ . In this case, the leaves of  $\mathcal{F}$  are covered by algebraic subvarieties of dimension  $n - k$ , the fibers of  $\pi$ .

Actually, the converse holds true. Suppose that through a general point of  $X$  there exists an algebraic subvariety tangent to  $\mathcal{F}$ . Since tangency to  $\mathcal{F}$  imposes a closed condition on the Hilbert scheme, it follows that the leaves of  $\mathcal{F}$  are covered by  $q$ -dimensional algebraic subvarieties,  $q = n - k$ . More precisely, there exists an irreducible algebraic variety  $Y$  and an irreducible subvariety  $Z \subset X \times Y$  such that the natural projections

$$\begin{array}{ccc} Z & \xrightarrow{\pi_2} & Y \\ \downarrow \pi_1 & & \\ X & & \end{array}$$

are both dominants; the general fiber of  $\pi_2$  has dimension  $q$ ; and the general fiber of  $\pi_2$  projects to  $X$  as a subvariety tangent to  $\mathcal{F}$ . By Stein factorization theorem, we can moreover assume that  $\pi_2$  has irreducible general fiber.



Throughout the paper we will make use of the following result. A particular version of it can be found in the proof of [34, Theorem 9.0.3].

**Lemma 2.4.** *Let  $\mathcal{F}$  be a foliation on a projective manifold  $X$  of dimension  $n$ . Assume that  $\mathcal{F}$  is covered by a family of  $(n - k)$ -dimensional algebraic subvarieties as above. Then  $\mathcal{F}$  is the pull-back of a foliation defined on a variety  $Y$  having dimension  $\leq k$ .*

*Proof.* When  $\pi_1 : Z \rightarrow X$  is birational, which means that through a general point passes exactly one subvariety  $Z_y$  of the family, then  $\pi_2 \circ \pi_1^{-1} : X \dashrightarrow Y$  is the pull-back map.

Suppose that our covering family  $Z \subset X \times Y$  is such that  $\dim(\pi_2^{-1}(y)) = q$  is maximal. If  $\pi_1$  is not birational then take a general point  $x \in X$  and let  $y \in Y$  be such that  $x \in Z_y = \pi_1 \pi_2^{-1}(y)$ . Then  $\pi_1 \pi_2^{-1} \pi_2 \pi_1^{-1}(Z_y)$  has dimension at least  $q + 1$  at  $x$  and is tangent to  $\mathcal{F}$  by construction, see [34, Lemma 9.1.6.1]. This contradicts the maximality of the dimension. The lemma follows.  $\square$

### 3. REDUCTION MODULO $p$

In this section we start our study of foliations with numerically trivial canonical bundle on projective manifolds. We are bound to restrict ourselves to the algebraic category as our results do depend on the reduction modulo  $p$  of foliations defined on complex projective manifolds.

**3.1. A few words about reduction modulo  $p$ .** Let  $\mathcal{F}$  be a foliation defined on a complex projective manifold  $X$ . The variety  $X$  and the subsheaf  $T\mathcal{F} \subset TX$ , can be both viewed as objects defined over a ring  $R$  of characteristic zero finitely generated over  $\mathbb{Z}$ . If  $\mathfrak{p} \subset R$  is a maximal ideal then  $R/\mathfrak{p}$  is a finite field  $k$  of characteristic  $p > 0$ . The reduction modulo  $\mathfrak{p}$  of  $\mathcal{F}$  is the *foliation*  $\mathcal{F}_{\mathfrak{p}}$  determined by the subsheaf  $T\mathcal{F}_{\mathfrak{p}} = T\mathcal{F} \otimes_R k$  of the tangent sheaf of the projective variety  $X_{\mathfrak{p}} = X \otimes_R k$ . In layman terms, we are just reducing modulo  $\mathfrak{p}$  the equations (which have coefficients in  $R$ ) defining  $X$  and  $\mathcal{F}$ . For more on the reduction modulo  $p$  see, [27] and [46, Chapter 1, §2.5].

Here we will use reduction modulo  $\mathfrak{p}$  to find invariant hypersurfaces and integrating factors for complex foliations with semi-stable tangent sheaves and numerically trivial canonical bundle. We will implicitly make use of the following result.

**Proposition 3.1.** *Let  $\mathcal{F}$  be a foliation on a polarized projective manifold  $(X, H)$  defined over a finitely generated  $\mathbb{Z}$ -algebra  $R \subset \mathbb{C}$ . If there are integers  $M, m$ , and a Zariski dense set of maximal primes  $\mathcal{P} \subset \text{Spec}(R)$  such that  $\mathcal{F}_{\mathfrak{p}}$  has an invariant subvariety of dimension  $m$  and degree at most  $M$  for every  $\mathfrak{p} \in \mathcal{P}$  then  $\mathcal{F}$  has an invariant hypersurface of dimension  $m$  degree at most  $M$ .*

*Proof.* For a fixed Hilbert polynomial  $\chi$ , the subschemes of  $X$  invariant by  $\mathcal{F}$  with Hilbert polynomial  $\chi$  form a closed subscheme  $\text{Hilb}_{\chi}(X, \mathcal{F})$  of  $\text{Hilb}_{\chi}(X)$ , see [21]. Moreover, its formation commutes with base change. Thus  $\text{Hilb}_{\chi}(X, \mathcal{F})$  is non-empty if and only if  $\text{Hilb}_{\chi}(X_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}})$  is non-empty for a Zariski dense set of primes  $\mathfrak{p}$ , see for instance [46, Lecture I, Proposition 2.6]. To conclude it suffices to remind that irreducible reduced subvarieties of  $X_{\mathfrak{p}}$  of bounded degree have bounded Hilbert polynomial, independently of  $\mathfrak{p}$ .  $\square$

If  $v$  is vector field on a smooth algebraic variety of positive characteristic then its  $p$ -th power is also a vector field since it satisfies Leibniz's rule:

$$v^p(f \cdot g) = \sum_{i=0}^p \binom{p}{i} v^i(f) v^{p-i}(g) = f v^p(g) + v^p(f) g \pmod{p}.$$

A foliation  $\mathcal{F}$  on a smooth algebraic variety  $X$  defined over a field of characteristic  $p > 0$  is said to be  $p$ -closed if and only if for every local section  $v$  of  $T\mathcal{F}$  its  $p$ -th power  $v^p$  is also a local section of  $T\mathcal{F}$ .

The  $p$ -closed foliations of codimension  $q$  are precisely those that can be defined by  $q$  rational functions  $f_1, \dots, f_q$  in the sense that  $df_1 \wedge \dots \wedge df_q$  is a non-zero rational section of  $\det N^*\mathcal{F}$  seen as a subsheaf of  $\Omega_X^q$ . Indeed, if  $\mathcal{F}$  is a  $p$ -closed foliation of codimension  $q$  then [46, Lecture III, 1.10] implies that at a general point of  $X$  there are local coordinates in which  $\mathcal{F}$  is defined by  $dx_1 \wedge \dots \wedge dx_q$ . Reciprocally, if  $\mathcal{F}$  is defined  $df_1 \wedge \dots \wedge df_q \neq 0$  then for every rational vector field  $v$  satisfying  $i_v df_1 \wedge \dots \wedge df_q = 0$  we have that

$$i_{v^p}(df_1 \wedge \dots \wedge df_q) = \sum_{i=1}^q (-1)^{i+1} v^p(f_i) \cdot df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge df_q = 0.$$

This illustrates what is perhaps the most astonishing contrast between foliations in positive/zero characteristic: the easiness/toughness to decide whether or not  $\mathcal{F}$  has first integrals.

If  $\mathcal{F}$  is a foliation on a projective manifold defined over a finitely generated  $\mathbb{Z}$ -algebra  $R \subset \mathbb{C}$  then the behavior of  $X_{\mathfrak{p}}$  and  $\mathcal{F}_{\mathfrak{p}}$  may vary wildly when  $\mathfrak{p}$  varies among the maximal primes of  $R$ . Thus in order to have some hope to read properties of  $\mathcal{F}$  on its reductions modulo  $\mathfrak{p}$  one has to discard the *bad primes*. When a foliation  $\mathcal{F}$  on a complex projective manifold has  $\mathfrak{p}$ -closed reduction modulo  $\mathfrak{p}$  for every maximal prime ideal  $\mathfrak{p}$  lying in a nonempty open subset  $U \subset \text{Spec}(R)$  then we will simply say that  $\mathcal{F}$  is  $p$ -closed.

As already mentioned in the Introduction, Ekedahl, Shepherd-Barron, and Taylor [27] conjectured that  $p$ -closed foliations are foliations by algebraic leaves. This generalizes a previous conjecture by Grothendieck and Katz about the reduction modulo  $p$  of flat connections. Despite the recent advances, notably [10], both conjectures are still wide open.

**3.2. Integrating factors in positive characteristic.** In this section we collect some results from [19, Section 6] which will be essential in what follows.

**Lemma 3.2.** *Let  $X$  be a smooth affine variety of dimension  $n$  defined over a field of arbitrary characteristic. If  $\omega$  is an integrable 1-form which is non zero at a closed point  $x \in X$  then there exists  $n - 1$  regular vector fields  $v_1, \dots, v_{n-1}$  at an affine neighborhood of  $x$  such that*

- (1)  $v_1 \wedge \dots \wedge v_{n-1}(x) \neq 0$ ;
- (2)  $[v_i, v_j] = 0$  for every  $i, j \in \{1, \dots, n-1\}$ ;
- (3)  $i_{v_i} \omega = 0$  for every  $i \in \{1, \dots, n-1\}$ .

*Proof.* This is lemma 6.1 from [19]. □

The underlying idea of the proof of the next result is that  $p$ -th powers of vector fields tangent to a integrable 1-form give rise to infinitesimal automorphisms and these allow us to find integrating factors. The proof presented below is borrowed

from the proof of [19, Theorem 6.2]. We have chosen to present it here since this result is pivotal in the proof of Theorem 3.5.

**Proposition 3.3.** *Let  $X$  be a smooth variety defined over a field  $k$  of characteristic  $p > 0$  and  $\omega$  be a rational 1-form on  $X$ . If  $\omega$  is integrable and there exists a rational vector field  $\xi$  such that*

- (1)  $i_\xi \omega = 0$ ; and
- (2)  $F = \omega(\xi^p) \neq 0$

*then the 1-form  $F^{-1} \cdot \omega$  is closed.*

*Proof.* Let  $n$  be the dimension of  $X$  and  $v_1, \dots, v_{n-1}$  be the rational vector fields given by Lemma 3.2. Thus  $\xi = \sum_{j=1}^{n-1} a_{ij} v_j$  for suitable rational functions  $a_{ij}$ . By a formula of Jacobson [32, page 187] we can write  $\xi^p = \sum_{j=1}^{n-1} a_{ij}^p v_j^p + P(a_{i,1}v_1, \dots, a_{i,n-1}v_{n-1})$  with  $P$  being a Lie polynomial. Since  $[v_i, v_j] = 0$  it follows that

$$\xi^p = \sum_{j=1}^{n-1} a_{ij}^p v_j^p, \quad \text{mod } \langle v_1, \dots, v_{n-1} \rangle.$$

As we are interested in contracting  $\xi^p$  with  $\omega$  we will replace  $\xi^p$  by  $\zeta = \sum_{j=1}^{n-1} a_{ij}^p v_j^p$ . Notice that  $[\zeta, v_j] = 0$  for  $j \in \{1, \dots, n-1\}$ .

Set  $\alpha = \frac{\omega}{\omega(\zeta)}$ . The integrability of  $\omega$  together with  $\alpha(\xi^p) = \alpha(\zeta) = 1$  implies

$$0 = i_\zeta(\alpha \wedge d\alpha) = d\alpha - \alpha \wedge i_\zeta d\alpha.$$

Hence to prove that  $\alpha$  is closed, it suffices to verify that the 1-form  $i_\zeta d\alpha$  is zero. As the vector fields  $v_1, \dots, v_{n-1}, \zeta$  commute, then for every vector field  $v$  in the previous list we have

$$(i_\zeta d\alpha)(v) = \alpha([\zeta, v]) - \zeta(\alpha(v)) + v(\alpha(\zeta)) = 0.$$

This ensures  $i_\zeta d\alpha = 0$ , and consequently  $d\alpha = 0$ . The proposition follows.  $\square$

**Corollary 3.4.** *Hypothesis as in Proposition 3.3. If  $\tilde{\xi}$  is another rational vector field satisfying (1) and (2) then the rational functions  $F = \omega(\xi^p)$  and  $\tilde{F} = \omega(\tilde{\xi}^p)$  differ by the multiplication of a  $p$ -th power of a rational function, i.e.,  $F = H^p \tilde{F}$ , for some rational function  $H$ . In particular, the identity  $\frac{dF}{F} = \frac{d\tilde{F}}{\tilde{F}}$  holds true.*

*Proof.* According to Proposition 3.3, both  $F^{-1}\omega$  and  $\tilde{F}^{-1}\omega$  are closed 1-forms. Therefore  $d(F^{-1}\tilde{F}) \wedge \omega = 0$ . Since the foliation defined by  $\omega$  is not  $p$ -closed, it follows that  $d(F^{-1}\tilde{F}) = 0$ . Hence  $F = H^p \tilde{F}$  for a suitable rational function  $H$ .  $\square$

**3.3. Lifting integrating factors.** We will now proceed to prove the main result of this Section.

**Theorem 3.5.** *Let  $(X, H)$  be a polarized projective complex manifold, and  $\mathcal{F}$  a semi-stable foliation of codimension one on  $X$ . If  $K\mathcal{F} \cdot H^{n-1} = 0$  then at least one of the following assertions holds true*

- (1) *the foliation  $\mathcal{F}$  is  $p$ -closed;*
- (2)  *$\mathcal{F}$  is induced by a closed rational 1-form with coefficients in a flat line bundle and without divisorial components in its zero set.*

*Proof.* Let  $R \subset \mathbb{C}$  be a finitely generated  $\mathbb{Z}$ -algebra such that everything in sight is defined over it. Suppose the set of maximal primes  $\mathcal{P} \subset \text{Spec}(R)$  for which  $\mathcal{F}_{\mathfrak{p}}$  – the reduction mod  $\mathfrak{p}$  of  $\mathcal{F}$  – is not  $p$ -closed is Zariski dense, and fix  $\mathfrak{p} \in \mathcal{P}$ .

To raise germs of vector fields  $v$  in  $T\mathcal{F}_{\mathfrak{p}}$  to their  $p$ -th powers provides a non-zero global section  $S_{\mathfrak{p}}$  of

$$\text{Hom}_{\mathcal{O}_X}(F^*T\mathcal{F}_{\mathfrak{p}}, N\mathcal{F}_{\mathfrak{p}}) = (F^*T\mathcal{F}_{\mathfrak{p}})^* \otimes N\mathcal{F}_{\mathfrak{p}}$$

where  $F$  is the absolute Frobenius.

Let us explicitly describe  $S_{\mathfrak{p}}$  at sufficiently small Zariski open subsets  $U_i$  disjoint from the singular set of  $\mathcal{F}_{\mathfrak{p}}$ . Let  $v_{1,i}, \dots, v_{n-1,i}$  be the  $n-1$  vector fields satisfying the conclusion of Lemma 3.2 and such that  $v_{1,i} \wedge \dots \wedge v_{n-1,i}$  does not vanish on  $U_i$ . We can also assume that  $\mathcal{F}_{\mathfrak{p}}$  is defined on the same domain by a 1-form  $\omega_i$  without divisorial components in its singular set. Take another open set  $U_j$  with the same properties. On overlapping charts, we have

$$\begin{pmatrix} v_{1,i} \\ \vdots \\ v_{n-1,i} \end{pmatrix} = M_{ij} \begin{pmatrix} v_{1,j} \\ \vdots \\ v_{n-1,j} \end{pmatrix}$$

where the matrix cocycle  $\{M_{ij}\}$  represents the cotangent bundle  $T^*\mathcal{F}_{\mathfrak{p}}$  of the foliation outside  $\text{sing}(\mathcal{F}_{\mathfrak{p}})$ .

As a consequence, using Jacobson's formula [32], we obtain

$$\begin{pmatrix} v_{1,i}^p \\ \vdots \\ v_{n-1,i}^p \end{pmatrix} = N_{ij} \begin{pmatrix} v_{1,j}^p \\ \vdots \\ v_{n-1,j}^p \end{pmatrix} \mod T\mathcal{F}_{\mathfrak{p}}$$

where the matrix  $N_{ij}$  is obtained from  $M_{ij}$  by replacing each entry by its  $p^{\text{th}}$  power. If we set  $s_{k,i} = \omega_i(v_{k,i}^p)$  then we gain the following equality

$$\begin{pmatrix} s_{1,i} \\ \vdots \\ s_{n-1,i} \end{pmatrix} = g_{ij} N_{ij} \begin{pmatrix} s_{1,j} \\ \vdots \\ s_{n-1,j} \end{pmatrix}$$

where  $g_{ij}$  is the cocycle representing the normal bundle of  $\mathcal{F}_{\mathfrak{p}}$ . The collection of vectors  $\{(s_{1,i}s_{2,i} \dots s_{n-1,i})^T\}$  represents  $S_{\mathfrak{p}}$  on  $X_{\mathfrak{p}} - \text{sing}(\mathcal{F}_{\mathfrak{p}})$ .

Let  $D_{\mathfrak{p}}$  be the zero divisor of the section  $S_{\mathfrak{p}}$ . Over  $U_i$ ,  $D_{\mathfrak{p}}$  is defined by the codimension one components of the common zeros of  $s_{1,i}, \dots, s_{n-1,i}$ . Since  $\mathcal{F}_{\mathfrak{p}}$  is not  $p$ -closed, there is at least one among these functions which do not vanish identically. Choose one for each open subset  $U_i$  and denote it by  $s_i$ . Corollary 3.4 guarantees that the zero divisor of two different choices will differ by an element in  $p \cdot \text{Div}(U_i)$ . It also implies that over nonempty intersections  $U_i \cap U_j$ ,  $s_i = g_{ij} h_{ij}^p s_j$  for some rational function  $h_{ij}$  in  $U_i \cap U_j$ . Therefore the rational 1-forms  $\frac{ds_i}{s_i}$  do not depend on the choices of the rational functions  $s_i$  and they satisfy

$$\frac{ds_i}{s_i} - \frac{ds_j}{s_j} = \frac{dg_{ij}}{g_{ij}}.$$

Notice also that the polar set of  $ds_i/s_i$  coincides with the irreducible components of  $D_{\mathfrak{p}|U_i}$  which have multiplicity relatively prime to  $p$ .

If we write  $g_{ij} = g_i/g_j$  as a quotient of rational functions on  $X$  then we can define on  $X_{\mathfrak{p}}$  a closed rational 1-form with simple poles  $\eta_{\mathfrak{p}}$  by setting

$$(\eta_{\mathfrak{p}})|_{U_i} = \frac{ds_i}{s_i} - \frac{dg_i}{g_i}$$

where we still denote by  $g_i$  the reduction modulo  $\mathfrak{p}$  of the rational functions  $g_i$ .

If  $C_{\mathfrak{p}}$  is an irreducible curve on  $X_{\mathfrak{p}}$  not contained in the polar set of  $\eta_{\mathfrak{p}}$  then the restriction of  $\eta_{\mathfrak{p}}$  to  $C_{\mathfrak{p}}$  is a rational 1-form with sum of residues equal to  $(D_{\mathfrak{p}} - N\mathcal{F}_{\mathfrak{p}}) \cdot C_{\mathfrak{p}} \pmod{p}$ . The residue formula implies the equality

$$(1) \quad D_{\mathfrak{p}} \cdot C_{\mathfrak{p}} = N\mathcal{F}_{\mathfrak{p}} \cdot C_{\mathfrak{p}} \pmod{p}.$$

If we set  $\omega = \omega_i/g_i$  then  $\omega$  is a well-defined rational 1-form on  $X$ . Moreover, Proposition 3.3 implies the identity  $d\omega = \omega \wedge \eta_{\mathfrak{p}}$  holds true on  $X_{\mathfrak{p}}$ . If we were in characteristic zero then the closed 1-form  $\eta_{\mathfrak{p}}$  would be the sought integrating factor.

Up to this point we have not used the hypothesis on  $K\mathcal{F}$ . In order to explore it and obtain further restrictions on  $D_{\mathfrak{p}}$ , we will use the following result by Shepherd-Barron, [53, Corollary 2<sup>p</sup>] and [34].

**Lemma 3.6.** (*char  $p$* ) *Suppose that  $\mathcal{E}$  is a semi-stable vector bundle of rank  $r$  over a curve  $C$  of genus  $g$ . Consider  $F^*\mathcal{E} = \tilde{\mathcal{E}}$ , the pull-back of  $\mathcal{E}$  under the absolute Frobenius, then there exists  $M = M(r, g) > 0$  independent of  $p$  such that*

$$\mu_{\max}(\tilde{\mathcal{E}}) - \mu_{\min}(\tilde{\mathcal{E}}) \leq M$$

Now, return to the original foliation  $\mathcal{F}$  on the complex manifold  $X$ . Consider a general complete intersection curve  $C$  cut out by elements of  $|mH|$  ( $m \gg 0$ ) for which the  $T\mathcal{F}|_C$  is semi-stable. Notice that this semi-stability is preserved under specialization  $\pmod{\mathfrak{p}}$  for almost every  $\mathfrak{p}$ .

Restricting  $S_{\mathfrak{p}}$  to  $C_{\mathfrak{p}}$  and cleaning up its zero divisor we get a section of

$$\mathrm{Hom}_{\mathcal{O}_{C_{\mathfrak{p}}}}(F^*\mathcal{F}_{\mathfrak{p}|C_{\mathfrak{p}}}, N\mathcal{F}_{\mathfrak{p}|C_{\mathfrak{p}}} \otimes \mathcal{O}_{C_{\mathfrak{p}}}(-D_{\mathfrak{p}})).$$

Since  $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{A}, \mathcal{B}) = 0$  whenever  $\mu_{\min}\mathcal{A} > \mu_{\max}\mathcal{B}$ , we deduce that

$$\mu_{\min}(F^*\mathcal{F}_{\mathfrak{p}|C_{\mathfrak{p}}}) \leq N\mathcal{F}_{\mathfrak{p}} \cdot C_{\mathfrak{p}} - D_{\mathfrak{p}} \cdot C_{\mathfrak{p}}.$$

Lemma 3.6 and the fact that  $\mu_{\max}(F^*\mathcal{F}_{\mathfrak{p}|C_{\mathfrak{p}}}) \geq 0$  implies

$$D_{\mathfrak{p}} \cdot C_{\mathfrak{p}} \leq M + N\mathcal{F}_{\mathfrak{p}} \cdot C_{\mathfrak{p}}$$

with  $M$  uniform in  $\mathfrak{p}$ . For  $p \gg 0$  this last inequality combined with (1) implies  $D_{\mathfrak{p}} \cdot C_{\mathfrak{p}} = N\mathcal{F}_{\mathfrak{p}} \cdot C_{\mathfrak{p}} = N\mathcal{F} \cdot C$ . Consequently the degree of  $D_{\mathfrak{p}}$  is uniformly bounded. In particular the polar locus of  $\eta_{\mathfrak{p}} + \frac{dg_i}{g_i}$  on  $U_i$  coincides with the support of  $D_{\mathfrak{p}}|_{U_i}$ . Thus we can lift  $D_{\mathfrak{p}}$  to a divisor  $D$  on  $X$ , and the 1-form  $\eta_{\mathfrak{p}}$  to a closed rational 1-form  $\eta$  with simple poles on  $X$  such that

$$d\omega = \omega \wedge \eta.$$

The residue of  $\eta + \frac{dg_i}{g_i}$  along an irreducible component  $H$  coincides with the multiplicity of  $H$  in  $D$ , and therefore lies in  $\mathbb{Z}_{>0}$ . We conclude that the 1-form

$$\frac{\omega}{\exp \int \eta}$$

is a closed rational 1-form on  $X$  with zero set of codimension at least two and coefficients in a flat line bundle defining the foliation  $\mathcal{F}$ .  $\square$

**3.4. Singularities of  $p$ -closed foliations.** McQuillan observed in [43, Proposition II.1.3] that isolated singularities of  $p$ -closed foliations of dimension one with non-nilpotent linear part are fairly special. The result below is a straightforward generalization of the proposition mentioned above. All the arguments used in its proof are already present in McQuillan's version.

**Lemma 3.7.** *Let  $\mathcal{F}$  be a  $p$ -closed foliation by curves on a projective manifold  $X$ . If  $x \in \text{sing}(\mathcal{F})$  is an isolated singularity with non-nilpotent linear part then there exist formal coordinates at  $x$  where  $\mathcal{F}$  is generated by the linear vector field*

$$v = \sum_{i=1}^n \lambda_i x_i \frac{\partial}{\partial x_i}$$

where  $\lambda_1, \dots, \lambda_n$  are non-zero integers.

*Proof.* In the terminology of [43] we have that an isolated singularity is log canonical if and only if its linear part is not nilpotent [43, Fact I.1.8]. The only case in our statement not covered by [43, Proposition II.1.3] is when the singularity is log canonical but not canonical. According to [43, Fact I.1.9] this implies that the linear part is diagonalizable and all the quotients of eigenvalues are positive rational numbers. If it is not formally linearizable then the Jordan-Chevalley decomposition [42] of a vector field  $v$  inducing  $\mathcal{F}$  can be written, in suitable formal coordinates, as

$$v = \underbrace{\sum_{i=1}^n m_i x_i \frac{\partial}{\partial x_i}}_{=v_S} + v_N$$

where  $m_1, \dots, m_n \in \mathbb{Z}_{>0}$  and  $v_N$  is vector field with zero linear part which commutes with  $v_S$ . These conditions imply that  $v_N$  is a finite linear combination of monomial vector fields of the form  $x^Q \frac{\partial}{\partial x_j} = x_1^{q_1} \dots x_n^{q_n} \frac{\partial}{\partial x_j}$  with  $Q = (q_1, \dots, q_n) \in \mathbb{Z}_{\geq 0}^n$  satisfying  $\sum_{i=1}^n q_i \geq 2$  and  $\sum_{i=1}^n q_i m_i = m_j$ .

Suppose  $v$  is not formally linearizable and let  $k$  be the degree of monomial vector field of smaller degree contributing to  $v_N$ . Then there are algebraic coordinates around  $x$  such that  $v = v_S + v_N^{(k)} \pmod{\mathfrak{m}_x^{k+1}}$  where

$$v_N^{(k)} = \sum_{j=1}^n \sum_{|Q|=k} a_{Q,j} x^Q \frac{\partial}{\partial x_j}$$

commutes with  $v_S$ .

Since our original data is algebraic, and we have only performed an algebraic change of coordinates we can assume that the coefficients of our original  $v$  and of the change of coordinates live in the same finitely generated  $\mathbb{Z}$ -algebra  $R$ . Now working modulo a maximal ideal  $\mathfrak{p} \subset R$  we see that  $v_S^p = \sum m_i^p x_i \frac{\partial}{\partial x_i} = v_S$  and Jacobson's formula implies

$$v^p = v_S + (v_N^{(k)})^p = v_S \pmod{\mathfrak{m}_x^{k+1}}.$$

Consequently the identity  $v \wedge v^p = v_N^{(k)} \wedge v_S \pmod{\mathfrak{m}_x^{k+2}}$  holds true.

The  $p$ -closedness of  $v$  implies  $v_N^{(k)} = P(x)v_S$  for a suitable homogeneous polynomial  $P$  of degree  $k-1 \geq 1$ . Since  $[v_S, v_N^{(k)}] = 0$ , the polynomial  $P$  must satisfy  $v_S(P) = 0$ . As the eigenvalues of  $v_S$ , in characteristic zero, are all positive integers

this identity cannot hold if the characteristic is sufficiently big. It follows that  $v$  is formally linearizable.  $\square$

**Theorem 3.8.** *Let  $(X, H)$  be a polarized complex projective manifold and  $\mathcal{F}$  be a codimension one semi-stable foliation on  $X$  with numerically trivial canonical bundle. Suppose  $c_1(TX)^2 \cdot H^{n-2} > 0$ . If  $\mathcal{F}$  is  $p$ -closed then*

- (1)  $\mathcal{F}$  is a rationally connected foliation, i.e., the general leaf of  $\mathcal{F}$  is a rationally connected algebraic variety; or
- (2)  $\mathcal{F}$  is strictly semi-stable and there is a rationally connected foliation  $\mathcal{H}$  tangent to  $\mathcal{F}$  and with  $K\mathcal{H} \cdot H^{n-1} = 0$ .

*Proof.* As  $c_1(T\mathcal{F}) = 0$ , we have that  $c_1(TX) = c_1(N\mathcal{F})$ . Thus  $c_1(TX)^2 \cdot H^{n-2} = c_1(N\mathcal{F})^2 \cdot H^{n-2} > 0$  and Baum-Bott index theorem [3] implies the existence of a codimension two component  $S$  of the singular set of  $\mathcal{F}$  which has positive Baum-Bott index.

Take a general surface  $\Sigma \subset X$  intersecting  $S$  transversally. Since  $\dim \Sigma = 2$  all the singularities of  $\mathcal{F}|_\Sigma$  are isolated. As  $p$ -closedness is preserved by restrictions to subvarieties, it follows from Lemma 3.7 that a singularity of  $\mathcal{F}|_\Sigma$  on  $S \cap \Sigma$  either has nilpotent linear part, or is formally linearizable with rational quotient of eigenvalues. Moreover, since the Baum-Bott index of  $S$  is positive, in the latter case the quotient of eigenvalues must be positive.

Assume first that every singularity of  $\mathcal{F}|_\Sigma$  on  $S \cap \Sigma$  have zero linear part. If we first resolve the singularities of  $S$  by means of blow-ups with smooth centers contained in its singular locus and then blow-up its strict transform, we obtain a birational morphism  $\pi : Y \rightarrow X$  such that the canonical bundle of  $\mathcal{G} = \pi^*\mathcal{F}$  is of the form

$$(2) \quad K\mathcal{G} = \pi^*K\mathcal{F} - E - D$$

here  $E$  is an effective divisor supported on an irreducible hypersurface such that  $\pi(|E|) = S$  and  $D$  is a divisor (not necessarily effective) such that  $\pi(|D|) \subset \text{sing}(S)$ . In particular  $\pi(|D|)$  has codimension at least three. We claim that the same holds true for any codimension two irreducible component of the singular set with positive Baum-Bott index. As we do not want to control the divisor  $D$ , there is no loss of generality if we assume  $S$  smooth.

If there is a singularity of  $\mathcal{F}|_\Sigma$  on  $S \cap \Sigma$  which has nonzero nilpotent linear part then [38, Corollary 3] (see also [20, Theorem 3.3]) implies that at a neighborhood of a general point of  $S$  there are coordinates  $(x_1, \dots, x_n)$  in which the foliation is defined by

$$\omega = x_1 dx_1 + (g \circ f + x_1 \cdot h \circ f) df$$

where  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$  and  $g, h : (\mathbb{C}, 0) \rightarrow \mathbb{C}$  are germs of holomorphic functions. Moreover,  $f$  does not depend on the first coordinate, i.e.  $\frac{\partial f}{\partial x_1} = 0$ . We can also assume, that  $S$  in this coordinate system is given by  $\{x_1 = x_2 = 0\}$ . It follows that  $f = x_2^k \tilde{f}$  for  $k \geq 1$  and a suitable germ  $\tilde{f} : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$  not divisible by  $x_2$ , see the proof of [20, Lemma 3.1]. There it is also proved that if  $\tilde{f}(0) = 0$  then  $k \geq 2$ ,  $g(0) \neq 0$ , and there exists a submersion  $H : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$  satisfying  $\frac{\partial H}{\partial x} = 0$ ,  $\frac{\partial^2 H}{\partial x^2} \neq 0$ , and  $\frac{\partial H}{\partial y} \neq 0$  such that  $H(x_1, f(x_2, \dots, x_n))$  is a holomorphic first integral for  $\omega$ . More explicitly, there exists a unity  $u \in \mathcal{O}_{\mathbb{C}^n, 0}^*$  such that

$$\omega = u(x_1, \dots, x_n) \cdot \left( \frac{\partial H}{\partial x}(x_1, f(x_2, \dots, x_n)) dx_1 + \frac{\partial H}{\partial y}(x_1, f(x_2, \dots, x_n)) df \right).$$

Notice that

$$H(x_1, f(x_2, \dots, x_n)) = x_1^2 + x_2^k \cdot \ell \mod \langle x_1^3, x_1 x_2^k, x_2^{k+1} \rangle$$

where  $\ell \in \mathbb{C}[x_1, x_2, \dots, x_n]$  is a linear form. If  $k = 2$  then at a general point of  $\{x_1 = x_2 = 0\}$  the restriction of  $\mathcal{F}$  to a general two-dimensional transversal is not nilpotent. Thus, under our hypothesis, the integer  $k$  is at least 3. If  $\pi(s, x_2, x_3, \dots, x_n) = (sx_2, x_2, x_3, \dots, x_n)$  is a local expression for the blow-up along  $S$  then

$$\pi^* \omega = \pi^* u \cdot x_2 \cdot (2s(sdx_2 + x_2 ds) + kx_2^{k-2} \ell dx_2 + x_2^{k-1} d\ell) + x_2 \cdot \eta,$$

with the coefficients of  $\eta$  in the ideal  $\langle s, x_2 \rangle^2$  since  $k \geq 3$ . It follows that  $N\pi^* \mathcal{F} = \pi^* N\mathcal{F} \otimes \mathcal{O}_{\tilde{X}}(-E)$ , and  $K\pi^* \mathcal{F} = \pi^* K\mathcal{F}$ . Notice that now the singular set of  $\pi^* \mathcal{F}$  contains a codimension two subvariety  $S_2 = \{s = x_2 = 0\}$  which has restriction to transverse sections with zero linear part. After resolving the singularities of  $S_2$  and blowing-up its strict transform we see that the claim follows.

Suppose now that  $\tilde{f}(0) \neq 0$ . According to the proof of [20, Lemma 3.1] there exists another system of coordinates  $(x, y, z = (z_1, \dots, z_{n-2}))$  in which

$$\omega = xdx + (g(y) + x \cdot h(y))dy$$

where  $g, h$  still denote germs of holomorphic functions vanishing at zero. Thus our problem reduces to dimension two. According to [12, page 13] the blow along  $S$  will give rise a codimension two component of the singular set contained in the exceptional divisor which either have zero linear part or non-zero but nilpotent linear part when restricted to a general two-dimensional transverse section. If we are in the former case we can conclude as before, and in the latter case (loc. cit.) one further blow-up will produce a codimension two component of the singular set with zero linear part at a general two-dimensional transverse section which also allow us to conclude as before.

If there is a singularity of  $\mathcal{F}|_{\Sigma}$  on  $S \cap \Sigma$  which is formally linearizable with linear part having rational positive quotient of eigenvalues then, since this singularity lies in the Poincaré domain [42] it is analytically linearizable. Kupka's Theorem (see for instance [25, Proposition 1.3.1]) implies that at a general point of  $S$  the foliation  $\mathcal{F}$  is defined in suitable holomorphic coordinates by

$$\omega = pydx - qxdy$$

with  $p, q$  relatively prime positive integers, according. If  $p = q = 1$  then the pull-back of  $\omega$  under the blow-up along  $S$  vanishes up to order two along the exceptional divisor. This implies the claim. If  $p \neq q$  then the singular set of the blow-up of  $\mathcal{F}$  along  $S$  has two irreducible components contained in the exceptional divisor. The new foliation will be defined at the general point of one of these irreducible components by  $(p - q)ydx - qxdy$  and at the other by  $pydx - (q - p)xdy$ . Notice that the canonical bundle of this new foliation is just the pull-back of the canonical bundle of the original one. If we iterate this procedure by blowing-up the irreducible component with positive quotient of eigenvalues, we eventually end up with an irreducible component of the singular set of the form  $xdy - ydx$ . One further blow-up proves the claim.

We have just proved the existence of a birational morphism  $\pi : Y \rightarrow X$  such that (2) holds true. Let  $A$  be an ample line bundle on  $Y$ . Let  $\varepsilon_0 > 0$  be such that  $K\mathcal{G} \cdot H_\varepsilon^{n-1} < 0$  for any positive  $\varepsilon \leq \varepsilon_0$ . If  $T\mathcal{G}$  is  $H_\varepsilon$ -semi-stable for some positive



$\varepsilon \leq \varepsilon_0$  then Corollary 2.3 implies that the leaves of  $\mathcal{G}$  are rationally connected varieties and we can conclude. If not then for any positive  $\varepsilon \leq \varepsilon_0$ , the maximal destabilizing foliation of  $\mathcal{G}$ , which we will denote by  $\mathcal{H}_\varepsilon$ , satisfies

$$(3) \quad \mu_\varepsilon(T\mathcal{H}_\varepsilon) > \mu_\varepsilon(T\mathcal{G})$$

where the slope  $\mu_\varepsilon$  is computed as a function of  $A$  and  $\varepsilon$ .

A priori, as  $\varepsilon$  goes to zero the maximal destabilizing foliation  $\mathcal{H}_\varepsilon$  could vary, but the proof of [47, Lemmma 3.3.3] shows that this cannot happen. More precisely, for  $\varepsilon > 0$  sufficiently small the maximal destabilizing foliations  $\mathcal{H}_\varepsilon$  will be all equal to a fixed foliation  $\mathcal{H}$ . Let us give the argument.

Let  $N_1(Y)$  be the space of 1-cycles in  $Y$  with real coefficients modulo numerical equivalence, and  $\overline{ME}(Y)$  the closed cone of (classes of) movable curves, as defined in [11]. Since  $H$  and  $A$  are ample divisors, the intersections  $(\pi^*H)^{n-1}, (\pi^*H)^{n-2} \cdot A, (\pi^*H)^{n-3} \cdot A^2, \dots, A^{n-1}$  define classes in  $\overline{ME}(Y) \subset N_1(Y)$ . We will denote by  $\Delta$  the convex cone generated by these classes. It is clearly closed, polyhedral, and generated by  $s \leq n$  elements which we will denote by  $C_1, \dots, C_s$ . Notice that for any  $\varepsilon \geq 0$  the intersection product  $(\pi^*H + \varepsilon A)^{n-1}$  defines an element of  $\Delta$ . Let  $D_1, \dots, D_\rho$  be generators of the Neron-Severi group of  $Y$  satisfying  $D_i \cdot C_j = \delta_{ij}$  for  $i, j \in \{1, \dots, s\}$ ; and  $D_i \cdot C_j = 0$  for  $i \in \{s+1, \dots, \rho\}$  and  $j \in \{1, \dots, s\}$ .

In [47], see also [15, Proposition 1.3], it is shown the existence of a Harder-Narasimhan filtration for reflexive sheaves where the slope is defined through the intersection with (classes of) movable curves instead of powers of ample divisors, i.e., for  $C \in \overline{ME}(Y)$  the slope of a reflexive sheaf  $\mathcal{E}$  with respect to  $C$  is defined as  $\mu_C(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cdot C}{\text{rank}(\mathcal{E})}$ . As the usual slope with respect to a polarization, the slope  $\mu_C$  defines a maximal destabilizing subsheaf  $\mathcal{E}_1 \subset \mathcal{E}$  maximizing the slope among all subsheaves of  $\mathcal{E}$ . In the particular case where  $C \in \overline{ME}(Y)$  is defined by the  $(n-1)$ -th power of an ample divisor  $D$ ,  $\mu_C$  coincides with  $\mu_D$ .

Suppose now that we have a sequence of  $\varepsilon_i$  of positive rational numbers converging to zero such that the foliations  $\mathcal{H}_{\varepsilon_i}$ , the maximal destabilizing foliation of  $\mathcal{G}$  with respect to the polarization  $H_{\varepsilon_i}$ , are all distinct. After passing to a subsequence we can assume that they all have the same dimension. We can write

$$c_1(T\mathcal{H}_{\varepsilon_i}) = \sum_{j=1}^s a_i^{(j)} D_j + R_i$$

where  $a_i^{(j)}$  are real numbers and  $R_i$  is a  $\mathbb{R}$ -divisor such that  $R_i \cdot H_\varepsilon^{n-1} = 0$  for every  $\varepsilon$ . Notice that  $\mu_{C_j}(T\mathcal{H}_{\varepsilon_i}) = a_i^{(j)}$  for  $j \in \{1, \dots, s\}$ . From the existence of maximal destabilizing subsheaves with respect to  $C_1, \dots, C_s$  it follows that the real numbers  $a_i^{(j)}$  are uniformly bounded from above. If they are not bounded from below, after passing to a subsequence we can assume that

$$c_1(T\mathcal{H}_{\varepsilon_i}) = \sum_{j=1}^t a_i^{(j)} D_j + \sum_{j=t+1}^s a^{(j)} D_j + R_i$$

where for any fixed  $1 \leq j \leq t$  the sequence  $a_i^{(j)}$  is strictly decreasing (and in particular tends to  $-\infty$  when  $i$  increases); and for  $j > t$  the coefficients of  $D_j$  are constant as a function of  $i$ . It follows that for  $k > 1$

$$\mu_{H_{\varepsilon_k}}(T\mathcal{H}_1) > \mu_{H_{\varepsilon_k}}(T\mathcal{H}_k),$$

which contradicts that  $T\mathcal{H}_k$  is the maximal destabilizing sheaf of  $T\mathcal{G}$  with respect to the polarization  $H_{\varepsilon_k}$ . Therefore the real numbers  $a_i^{(j)}$  have uniformly bounded norm, and after passing again to a subsequence we can assume that they do not depend on  $i$ . But then we deduce that  $\mu_{H_\varepsilon}(T\mathcal{H}_{\varepsilon_i}) = \mu_{H_\varepsilon}(T\mathcal{H}_{\varepsilon_j})$  for any pair  $i, j$  in our subsequence contradicting the uniqueness of the maximal destabilizing subsheaf of  $T\mathcal{G}$  with respect to the polarizations  $H_{\varepsilon_i}$  and  $H_{\varepsilon_j}$ . The existence a common maximal destabilizing foliation  $\mathcal{H}$  for the polarizations  $H_\varepsilon$  with sufficiently small positive  $\varepsilon$  follows. Since  $\mu_{H_{\varepsilon_i}}(T\mathcal{H}) > \mu_{H_{\varepsilon_i}}(T\mathcal{G}) > 0$ , Corollary 2.3 implies that the generic leaf of  $\mathcal{H}$  is rationally connected.

To conclude notice that on the one hand  $\mu_{\varepsilon_k}(T\mathcal{H}) > \mu_{\varepsilon_k}(T\mathcal{G}) > 0$  implies that  $\mu_{\pi^*H}(T\mathcal{H}) = \mu_H(T\pi_*\mathcal{H}) \geq 0$ . On the other hand, the  $H$ -semistability of  $T\mathcal{F} = T\pi_*\mathcal{G}$  implies  $\mu_H(T\pi_*\mathcal{H}) \leq 0$ . It follows that  $\mu_H(T\pi_*\mathcal{H}) = 0$ . Consequently  $T\mathcal{F}$  is strictly semi-stable, and  $\pi_*\mathcal{H}$  is the sought foliation tangent to  $\mathcal{F}$  with rationally connected general leaf.  $\square$

Theorem 3.8 plays an essential role in the classification of codimension foliations with trivial canonical bundle on Fano 3-folds with Picard number one, see [41].

We could have simplified the proof of Theorem 3.8 considerably by using the computations of the Baum-Bott indexes of the nilpotent components carried out in [20]. The more *complicated* prove presented here has the advantage of implying the following result, which will be used in the proof of Theorem 1.

**Corollary 3.9.** *Let  $\mathcal{F}$  be a  $p$ -closed foliation with  $c_1(K\mathcal{F}) = 0$  on a projective manifold  $X$ . If  $\mathcal{F}$  is not uniruled and  $S$  is an irreducible component of  $\text{sing}(\mathcal{F})$  of codimension two then at a general point of  $S$  the foliation  $\mathcal{F}$  is locally defined by a holomorphic 1-form of type*

$$pxdy + qydx$$

*with  $p, q$  relatively prime positive integers.*

Later in Section 4 we will show that under the same hypothesis of the corollary above we can either ensure the existence of at least one irreducible component of  $\text{sing}(\mathcal{F})$  having  $(p, q) \neq (1, 1)$ , or the foliation is smooth.

#### 4. CRITERIUM FOR UNIRULEDNESS

We now turn back to the problem of describing the structure of an arbitrary codimension one foliation with numerically trivial canonical bundle. The main goal of this section is to obtain information about the ambient manifold when the singular set of  $\mathcal{F}$  is not empty.

**4.1. Pseudo-effectiveness of the canonical bundle implies smoothness.** A particular case of the result below ( $p = \dim X - 1$ ) already appeared in [58]. The arguments here are a simple generalization of the arguments therein. They have high order of contact with the arguments carried out by Bogomolov in [7], see also [48, Section 5].

**Theorem 4.1** (First part of Theorem 4 of the Introduction). *Let  $X$  be a compact Kähler manifold with  $KX$  pseudo-effective,  $L$  be a flat line bundle on  $X$ ,  $p$  a positive integer and  $v \in H^0(X, \bigwedge^p TX \otimes L)$  a non-zero section; then the zero set of  $v$  is empty.*

*Proof.* Let  $\{U_i\}$  be an open covering of  $X$ ,  $\Omega_i \in KX(U_i)$  be holomorphic  $n$ -forms trivializing  $KX$ . If we write  $\Omega_i = g_{ij}^{-1} \Omega_j$  then  $\{g_{ij}\}$  is a cocycle defining  $KX$ . Let  $\omega_i$  be the contraction of  $v$  with the  $n$ -form  $\Omega_i$ . Notice that the collection  $\{\omega_i\}$  defines a holomorphic section of  $\Omega_X^q \otimes KX^* \otimes L$  with  $q = n - p$ .

The pseudo-effectiveness of  $KX$  implies the existence of singular hermitian metric on it with non-negative curvature. The same holds for  $KX \otimes L^*$  by flatness. Concretely, there exists plurisubharmonic functions  $\varphi_i$  on  $U_i$  such that

$$|h_{ij}|^2 = \exp(\varphi_i - \varphi_j).$$

where  $h_{ij}$  is a cocycle defining  $KX \otimes L^*$ . Thus the  $(q, q)$ -form

$$\eta = \sqrt{-1} \exp(\varphi_i) \omega_i \wedge \bar{\omega}_i$$

is a global well-defined real  $(q, q)$ -form with coefficients in  $L_{loc}^\infty$ . Demailly, in [24], proved that the identity of currents  $d\omega_i = -\partial\varphi_i \wedge \omega_i$  holds true, see also the proof of [14, Proposition 2.1]. Consequently,  $d\eta = 0$  as a current, and  $\eta$  defines a class in  $H^{q,q}(X, \mathbb{R})$ . By Poincaré Serre duality, there exists  $[\beta] \in H^{n-q, n-q}(X, \mathbb{R})$  such that  $[\eta] \wedge [\beta] \neq 0$ .

Decompose  $\eta$  as the product  $(-\sqrt{-1} \exp(\varphi_i) \bar{\omega}_i) \wedge (\omega_i)$ . From

$$\exp(\varphi_i) \bar{\omega}_i = |h_{ij}|^2 \exp(\varphi_j) \bar{h}_{ij}^{-1} \bar{\omega}_j = h_{ij} \exp(\varphi_j) \bar{\omega}_j.$$

we deduce that the first factor of  $\eta$  is a  $\bar{\partial}$ -closed  $(0, q)$ -form with values in  $KX \otimes L^*$ . Similarly the second factor of  $\eta$  is a holomorphic  $q$ -form with values in  $KX^* \otimes L$ . Notice that

$$0 \neq [\sqrt{-1} \exp(\varphi_i) \bar{\omega}_i] \wedge [\omega_i \wedge \beta]$$

and that we can interpret the first factor as a class in  $H_{\bar{\partial}}^{0,q}(KX \otimes L^*)$ , and the second factor as a non trivial class in  $H_{\bar{\partial}}^{n,n-q}(KX^* \otimes L) = H_{\bar{\partial}}^{0,n-q}(L)$ .

Let  $\gamma$  be a harmonic representative of  $[\omega_i \wedge \beta]$  in  $H_{\bar{\partial}}^{0,n-q}(L)$  (where  $L$  is endowed with a flat hermitian metric). Hodge symmetry implies  $\bar{\gamma}$  is a holomorphic  $(n - q)$ -form valued in  $L^*$ . Therefore

$$[\sqrt{-1} \exp(\varphi_i) \bar{\omega}_i \wedge \gamma] \neq 0$$

and consequently  $\{\bar{\gamma} \wedge \omega_i\}$  is a non-zero section of  $KX \otimes KX^* \otimes L \otimes L^* = \mathcal{O}_X$ . It follows that  $\omega_i$  has no zeros, and the same holds for  $v$ .  $\square$

**Theorem 4.2** (Second part of Theorem 4 of the Introduction). *Let  $\mathcal{D}$  be a distribution of codimension  $q$  on a compact Kähler manifold  $X$ . If  $c_1(T\mathcal{D}) = 0$  and  $KX$  is pseudoeffective then  $\mathcal{D}$  is a smooth foliation. Moreover, there exists a smooth foliation  $\mathcal{G}$  on  $X$  of dimension  $q$  such that  $TX = T\mathcal{D} \oplus T\mathcal{G}$ . Finally, if  $X$  is projective, then the canonical bundle of  $\mathcal{D}$  is torsion.*

*Proof.* The integrability follows from [24]. The previous theorem implies that  $\text{sing}(\mathcal{D}) = \emptyset$  and that there exists a holomorphic  $(n - q)$ -form  $\bar{\gamma}$  which restricts to a volume form on the leaves of the foliation defined by  $\mathcal{D}$ .

In order to prove the result we just need to modify  $\bar{\gamma}$  to obtain that its kernel is the expected complementary subbundle defining  $\mathcal{G}$ . This can be done as follows. There is a natural monomorphism of sheaves

$$\psi : \bigwedge^{n-q-1} T\mathcal{D} \rightarrow \Omega_X^1,$$

defined by the contraction of  $\bar{\gamma}$  with  $n-q-1$  vectors fields tangent to  $\mathcal{D}$ . Notice that the projection morphism of  $\Omega_X^1$  onto  $T^*\mathcal{D}$  is actually an isomorphism in restriction to  $Im\ \psi$ . Its inverse provides a splitting of the exact sequence

$$0 \rightarrow N^*\mathcal{D} \rightarrow \Omega_X^1 \rightarrow T^*\mathcal{D} \rightarrow 0.$$

Since  $\det T^*\mathcal{D}$  is numerically trivial,  $Im\ \psi$  is an integrable subbundle of  $\Omega_X^1$ . This subbundle defines the conormal bundle of the sought foliation  $\mathcal{G}$ .

Let  $L = K\mathcal{D}^*$  and  $\gamma \in H^0(X, \Omega_X^p \otimes L)$ ,  $p = n - q$ , be a twisted  $p$ -form defining  $\mathcal{G}$ . After passing to a finite étale covering we can assume that the integral Chern class of  $L$  is zero, i.e.,  $L \in \text{Pic}^0(X)$ .

Since  $L$  is flat, Hodge symmetry implies that  $H^0(X, \Omega_X^p \otimes L) \cong H^p(X, L^*)$ . Let  $m = h^p(X, L^*)$  and consider the Green-Lazarsfeld set

$$S = \{E \in \text{Pic}^0(X) \mid h^p(X, E) \geq m\}.$$

According to [54], if  $X$  is projective then  $S$  is a finite union of translates of subtori by torsion points. To conclude the proof of the Theorem it suffices to show that  $L^*$  is an isolated point of  $S$ . Let  $\Sigma \subset \text{Pic}^0(X)$  be an irreducible component of  $S$  passing through  $L$ . If  $\mathcal{P}$  is the restriction of the Poincaré bundle to  $\Sigma \times X$  and  $\pi : \Sigma \times X \rightarrow \Sigma$  is the natural projection then, by semi-continuity,  $R^p\pi_*\mathcal{P}$  is locally free at a neighborhood of  $L$ . Therefore we can extend the element  $H^p(X, L^*)$  determined by  $\gamma$  to a holomorphic family of non-zero elements with coefficients in line bundles  $E \in \Sigma$  close to  $L^*$ . Hodge symmetry gives us a family of holomorphic  $p$ -forms with coefficients in the duals of these line bundles. Taking the wedge product of these  $p$ -forms with a  $q$ -form defining  $\mathcal{D}$  we obtain, by transversality of  $\mathcal{D}$  and  $\mathcal{G}$ , non-zero sections of  $H^0(X, KX \otimes N\mathcal{D} \otimes E^*)$  for  $E$  varying on a small neighborhood of  $L^*$  at  $\Sigma$ . Since  $KX \otimes N\mathcal{D} \otimes E^* \in \text{Pic}^0(X)$ , this implies that  $E \in \Sigma$  if and only if  $E = KX \otimes N\mathcal{D} = L^*$ . Thus  $\Sigma$  reduces to a point.  $\square$

**Remark 4.3.** In the same vein, one can prove that the flat line bundle  $L$  in theorem 4.1 is actually a torsion one (under the extra assumption that  $X$  is projective). The Theorem above provides evidence toward the following conjecture of Sommese ([55]): if  $\mathcal{F}$  is a smooth foliation of dimension  $p$  with trivial canonical bundle on a compact Kähler manifold  $X$  then there exists a holomorphic  $p$ -form on  $X$  which is non-trivial when restricted to the leaves of  $\mathcal{F}$ .

Theorem 4.2 reduces the classification of codimension one foliations with numerically trivial canonical bundle on Kähler manifolds with pseudo-effective canonical bundle to the work done by the third author in [56] and recalled in the Introduction. The case of smooth foliations of higher codimension with numerically trivial canonical bundle on compact Kähler manifolds have also been treated by the third author in [57], but the results are not as complete as in the codimension one case. As shown by Peternell in his recent preprint ([49]) already mentioned in the introduction, such foliations arise naturally, at least on non uniruled projective manifolds, when the cotangent bundle fails to be generically ample. We redirect the interested reader to the above mentioned papers.

**4.2. Criterium for uniruledness.** The structure of projective manifolds carrying non-zero holomorphic vector fields are fairly well understood, see for instance [1, Theorem 0.1]. In particular, Rosenlicht proved that projective manifolds having a non-zero vector field with non-empty zero set are uniruled, and Lieberman later

generalized this result to the Kähler realm (loc. cit. Section 1). More recently, Campana and Peternell proved the following result [15, Corollary 1.12]: if the  $m$ -th tensor power  $TX^{\otimes m}$  of the tangent sheaf of a projective manifold  $X$  admits a subsheaf  $\mathcal{E}$  of rank  $r$  such that  $\det \mathcal{E}$  is pseudo-effective and the induced section  $\wedge^r TX^{\otimes m} \otimes \det \mathcal{E}^*$  vanishes along a divisor then  $X$  is uniruled. Theorem 4.1 allow us to deduce a strictly related result which confirms [49, Conjecture 4.23].

**Theorem 4.4** (Theorem 3 of the Introduction). *Let  $X$  be a projective manifold and  $L$  be a pseudo-effective line bundle on  $X$ . If there exists non trivial  $v \in H^0(X, \wedge^p TX \otimes L^*)$  vanishing at some point then  $X$  is uniruled. In particular, if there exists a foliation  $\mathcal{F}$  on  $X$  with  $c_1(T\mathcal{F})$  pseudo-effective and  $\text{sing}(\mathcal{F}) \neq \emptyset$  then  $X$  is uniruled.*

*Proof.* If  $X$  is not uniruled then  $KX$  is pseudo-effective [11, Corollary 0.3]. Miyaoka's Theorem (Corollary 2.3) together with Mehta-Ramanathan Theorem [44] imply that the Harder-Narasimhan filtration of the restriction of  $TX$  to curves obtained as complete intersections of sufficiently ample divisors has no subsheaf of positive degree. Therefore the same holds true for  $\wedge^p TX$ , and consequently  $L$  cannot intersect ample divisors positively. This property together with its pseudo-effectiveness implies  $c_1(L) = 0$ . We can apply Theorem 4.1 to conclude that  $\text{sing}(v) = \emptyset$ , and obtain a contradiction.  $\square$

#### 4.3. Division property implies smoothness.

**Proposition 4.5.** *Let  $\mathcal{F}$  be a codimension one foliation with numerically trivial canonical bundle on a Kähler manifold  $X$ . If  $h^1(X, N^*\mathcal{F}) \neq 0$  then  $\mathcal{F}$  is smooth or there exists a foliation  $\mathcal{G}$  by rational curves tangent to  $\mathcal{F}$ .*

*Proof.* Let  $\theta \in H^1(X, N^*\mathcal{F})$  be a non-zero element. Serre duality produces a non-zero element in  $H^{n-1}(X, KX \otimes N\mathcal{F}) \cong H^{n-1}(X, K\mathcal{F})$ . Since  $K\mathcal{F}$  is numerically trivial by hypothesis, we can apply Hodge symmetry to obtain a non-zero element  $v_\theta \in H^0(X, \Omega_X^{n-1} \otimes K^*\mathcal{F}) \cong H^0(X, TX \otimes N^*\mathcal{F})$ . Contract  $v_\theta$  with a twisted 1-form  $\omega \in H^0(X, \Omega_X^1 \otimes N\mathcal{F})$  defining  $\mathcal{F}$  to obtain a section of  $\mathcal{O}_X$ . It is either nowhere zero or identically zero. In the first case  $\mathcal{F}$  must be smooth and in the second we have a foliation by curves  $\mathcal{G}$  tangent to  $\mathcal{F}$  with canonical bundle  $K\mathcal{G} = N^*\mathcal{F} - \Delta$ , where  $\Delta$  is the divisor of zeros of  $v_\theta$ . If  $K\mathcal{G}$  is pseudo-effective then the same is true for  $N^*\mathcal{F} = KX$  (numerically) and we can apply Theorem 4.1 to conclude. If  $K\mathcal{G}$  is not pseudo-effective then  $\mathcal{G}$  must be a foliation by rational curves according to Brunella's Theorem [13] (if  $X$  is projective it suffices to invoke Miyaoka's Theorem).  $\square$

**Theorem 4.6.** *Let  $\mathcal{F}$  be a codimension one foliation with numerically trivial canonical bundle on a Kähler manifold  $X$  defined by  $\omega \in H^0(X, \Omega_X^1 \otimes N\mathcal{F})$ . If  $\Sigma \subset X$  is an analytic subset of codimension at least three, and for every point  $x \in X - \Sigma$  there exists a germ of holomorphic 1-form  $\eta$  at  $x$  such that  $d\omega = \eta \wedge \omega$  holds in a neighborhood of  $x$  then  $\mathcal{F}$  is smooth.*

*Proof.* Since  $K\mathcal{F}$  is numerically zero, the canonical bundle  $KX$  is numerically equivalent to  $N^*\mathcal{F}$ .

Let  $\{U_i\}$  be an open covering of  $X$  by Stein open subsets and  $\omega_i \in \Omega_X^1(U_i)$  be local representatives of  $\omega$ . At the intersections  $\omega_i = g_{ij}\omega_j$ , where  $\{g_{ij}\}$  is a cocycle defining  $N\mathcal{F}$  in  $H^1(X, \mathcal{O}_X^*)$ . Notice that the logarithmic derivative  $\frac{dg_{ij}}{g_{ij}}$  represents

(up to a constant factor) the Chern class of  $N^*\mathcal{F}$  in  $H^1(X, \Omega_X^1)$ . If  $c_1(N^*\mathcal{F}) = 0$  then  $KX$  is numerically equivalent to zero; in particular, it is pseudo-effective, and we can thus apply Theorem 4.1 to deduce the smoothness of  $\mathcal{F}$ . From now on, we will assume that  $N^*\mathcal{F}$  has non-zero Chern class.

We will now use the division property to obtain another representative in  $H^1(X, \Omega_X^1)$  for  $c_1(N^*\mathcal{F})$ .

We claim that there exists  $\eta_i \in \Omega_X^1(U_i)$  satisfying  $d\omega_i = \eta_i \wedge \omega_i$ . For a fixed  $i$ , let  $\{V_j\}$  be an open covering of  $U_i - \Sigma$  such that for every  $j$  there exists  $\eta_{ij}$  satisfying  $d\omega_i = \eta_{ij} \wedge \omega_i$ . At non-empty intersections  $V_j \cap V_k$ , there exists  $h_{jk} \in \mathcal{O}(V_j \cap V_k)$  such that  $\eta_{ij} - \eta_{ik} = h_{jk}\omega_i$ . Since  $U_i$  is Stein,  $H^1(U_i, \mathcal{O}_{U_i}) = 0$ . As  $\Sigma$  has codimension at least three,  $H^1(U_i - \Sigma, \mathcal{O}_{U_i - \Sigma})$  also vanishes [29, pg. 133]. Thus we can write  $h_{jk} = h_j - h_k$  with  $h_j \in \mathcal{O}(V_j)$  and  $h_k \in \mathcal{O}(V_k)$ . Patching together the 1-forms  $\eta_{ij} - h_j\omega_i \in \Omega^1(V_j)$  we obtain a 1-form  $\eta_i \in \Omega^1(U_i - \Sigma)$  satisfying  $d\omega_i = \eta_i \wedge \omega_i$  on  $U_i - \Sigma$ . Hartog's extension Theorem allow us to extend  $\eta_i$  to the whole  $U_i$ , proving the claim.

A simple computation shows that the 1-forms

$$\theta_{ij} = \frac{dg_{ij}}{g_{ij}} - (\eta_i - \eta_j)$$

vanish along the leaves of  $\mathcal{F}$ . Therefore the collection  $\{\theta_{ij}\}$  defines an element  $\theta \in H^1(X, N^*\mathcal{F})$  with image in  $H^1(X, \Omega_X^1)$  equal to the first Chern class of  $N^*\mathcal{F}$ , in particular  $h^1(X, N^*\mathcal{F}) \neq 0$ . We can apply Proposition 4.5 to ensure that  $\mathcal{F}$  is smooth, or there exists a foliation by rational curves  $\mathcal{G}$  tangent to  $\mathcal{F}$ . To conclude the proof it remains to exclude the latter possibility.

Let  $i : \mathbb{P}^1 \rightarrow X$  be a generically injective morphism to a general leaf of  $\mathcal{G}$ . Since it is tangent to  $\mathcal{F}$ , then the 1-forms  $i^*\theta_{ij}$  vanish identically. Therefore  $i^*N^*\mathcal{F}$  is the trivial line bundle. Consequently  $i^*KX = i^*K\mathcal{F} \otimes i^*N^*\mathcal{F}$  is also trivial. On the other hand, since  $i(\mathbb{P}^1)$  moves in a family of rational curves which cover  $X$ ,

$$i^*TX = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$$

with  $a_1 \geq 2$  and  $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ . Thus  $i^*KX = \mathcal{O}_{\mathbb{P}^1}(-\sum a_i)$  contradicting its triviality. Therefore the foliation by rational curves  $\mathcal{G}$  cannot be contained in  $\mathcal{F}$ .  $\square$

**Corollary 4.7.** *Let  $\mathcal{F}$  be a foliation of codimension one with  $c_1(K\mathcal{F}) = 0$  on a projective manifold  $X$ . If  $\text{sing}(\mathcal{F}) \neq \emptyset$  then it has an irreducible component of codimension two.*

**Remark 4.8.** Among the examples of codimension one foliations with  $c_1(K\mathcal{F}) = 0$ , one finds the foliations defined by Poisson structures of corank one. Corollary 4.7 implies that these either have constant rank or that the rank drops in a codimension two subvariety. Therefore it generalizes and gives a conceptual proof of [5, Proposition 4 item 3] as asked by Beauville. It is also in accordance with Bondal's conjecture [5, Conjecture 4].

Combining Corollaries 3.9 and 4.7 we obtain the following result which will be useful later.

**Corollary 4.9.** *Let  $\mathcal{F}$  be a  $p$ -closed codimension one foliation with  $c_1(K\mathcal{F}) = 0$  on a projective manifold  $X$ . If  $\mathcal{F}$  is not uniruled and  $\text{sing}(\mathcal{F}) \neq \emptyset$  then there exists an*

irreducible component  $S$  of  $\text{sing}(\mathcal{F})$  having codimension two and at a general point of  $S$  the foliation  $\mathcal{F}$  is locally defined by a holomorphic 1-form of type

$$pxdy + qydx$$

with  $p, q$  relatively prime **distinct** positive integers.

*Proof.* Since  $\mathcal{F}$  is not uniruled by hypothesis, Corollary 3.9 implies that at a general point of every codimension two irreducible component of the singular set of  $\mathcal{F}$ , the foliation is defined by a 1-form of the type  $pxdy + qydx$  with  $p, q$  relative prime positive integers. If  $p = q$  then  $\mathcal{F}$  is locally defined by the closed 1-form  $d(xy)$ . If this holds for every codimension two irreducible component of  $\text{sing}(\mathcal{F})$  then  $\mathcal{F}$  satisfies the hypothesis of Theorem 4.6 and therefore is smooth. This contradicts  $\text{sing}(\mathcal{F}) \neq \emptyset$  and proves the corollary.  $\square$

## 5. PROJECTIVE STRUCTURES AND TRANSVERSELY PROJECTIVE FOLIATIONS

**5.1. Rational projective structures on  $\mathbb{P}^1$ .** On a Riemann surface, a projective structure is defined by an atlas taking values in  $\mathbb{P}^1$  with transition charts in  $\text{Aut}(\mathbb{P}^1) = \text{PGL}(2, \mathbb{C})$ . On  $\mathbb{P}^1$  there is only one projective structure. Therefore it is natural to allow meromorphic singularities. There are at least two equivalent ways to define a meromorphic projective structure on  $\mathbb{P}^1$ . One can first define it by a rational quadratic form  $\eta$ . The poles are the singular points of the structure. Given a *linear* coordinate  $x$ , one can write  $\eta = \phi(x)dx^2$  and the charts  $\varphi$  of the structure are the solutions of the equation

$$\{\varphi, x\} = \phi$$

where  $\{\varphi, x\}$  stands for the Schwarzian derivative of  $\varphi$  with respect to  $x$ :

$$\{\varphi, x\} = \left( \frac{\varphi''}{\varphi'} \right)' - \frac{1}{2} \left( \frac{\varphi''}{\varphi'} \right)^2.$$

This is the classical approach which can be traced back to Schwarz and Poincaré, [51, Chapter IX].

One can also define the structure by the data of a  $\mathbb{P}^1$ -bundle  $P \rightarrow \mathbb{P}^1$ , a Riccati foliation  $\mathcal{H}$  on  $P$  (i.e. a singular foliation on the ruled surface  $P$  which is transversal to a general fibre of the ruling) and a section  $\sigma : \mathbb{P}^1 \rightarrow P$  which is not  $\mathcal{H}$ -invariant. At the neighborhood of a fibre transversal to  $\mathcal{H}$ , one can find coordinates  $(x, z)$  such that  $x$  defines the ruling, and  $z \in \mathbb{P}^1$  defines the foliation. Then  $\varphi := z \circ \sigma$  is a chart of the structure. If  $(P', \mathcal{H}', \sigma')$  is derived in the natural way by a birational modification  $P \dashrightarrow P'$  of the bundle, it then defines the same projective structure. From this point view, a singular projective structure is the data of the triple  $(P, \mathcal{H}, \sigma)$  up to birational modification. There is however a representative which is well-defined up to bundle isomorphisms. It is characterized by the following conditions :

- (1) the (effective) polar divisor has minimal degree,
- (2) the section  $\sigma$  does not intersect the singular locus of  $\mathcal{H}$ .

This is the minimal model discussed in [39]. Condition (1) characterizes the relative minimal model for  $(P, \mathcal{H})$  discussed in [12, Chapter 5]; there are countably many except in some particular cases and condition (2) fixes this freedom.

The equivalence between the two point of view described above is as follows. Given the quadratic form  $\eta$  defining the projective structure, one can associate the

foliation  $\mathcal{H}$  defined on the trivial bundle  $\mathbb{P}^1 \times \mathbb{P}^1$  with projective coordinates  $(x, z)$  by the Riccati equation  $\frac{dz}{dx} + z^2 + \frac{\phi(x)}{2} = 0$  and the section  $z = \infty$ .

Conversely, given a triple  $(P, \mathcal{H}, \sigma)$ , one can assume, up to birational bundle transformations, that  $P$  is the trivial bundle with coordinate  $z$  and moreover  $\sigma$  is defined by  $z = \infty$ . The foliation  $\mathcal{H}$  is thus defined by a 1-form

$$\Omega = dz + (f(x)z^2 + g(x)z + h(x))dx.$$

Using a birational map of the form  $(x, z) \mapsto (x, a(x)z + b(x))$ , we are able to suppose that  $f(x) = 1$  and  $g(x) = 0$ . Precisely, if

$$(4) \quad \vartheta : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

$$(x, z) \mapsto \left( x, \frac{2z + \frac{f'}{f} - g}{2f} \right)$$

then

$$\vartheta^* \Omega = \frac{1}{f} \left( dz + \left( z^2 + \frac{\phi(x)}{2} \right) dx \right)$$

where

$$(5) \quad \phi = \{f, x\} + 2fh - g' + g \frac{f'}{f} - \frac{1}{2}g^2.$$

One easily check that the relative minimal models of singular fibres described in [12, p. 52-56] determine the order of poles of  $\eta$  at a singular fibre of  $\mathcal{H}$ . The only other poles of  $\eta$  are double, and come from the tangency points of  $\sigma$  with  $\mathcal{H}$ .

For a more thorough account of the material presented here see [40].

**5.2. Transversely projective, affine, and Euclidean foliations.** We now turn to transversely projective foliations following [52, 18, 19, 39]. A *transversely projective* structure for a codimension one foliation  $\mathcal{F}$  on a projective manifold  $X$  is the data  $(P, \mathcal{H}, \sigma)$  of a  $\mathbb{P}^1$ -bundle  $P \rightarrow X$ , a Riccati foliation  $\mathcal{H}$  on  $P$ , and a meromorphic section  $\sigma : X \dashrightarrow P$  such that  $\sigma^* \mathcal{H} = \mathcal{F}$ .

Another triple  $(P', \mathcal{H}', \sigma')$  defines the same transversely projective structure if it is derived from the initial one by a birational bundle transformation  $P \dashrightarrow P'$ . Up to such birational bundle transformation, one can always assume that  $P$  is the trivial bundle  $X \times \mathbb{P}^1$  with vertical coordinate  $z$  and  $\sigma$  is the section  $\{z = 0\}$ . Notice we have switched from the section at infinity to the section at zero. The foliation  $\mathcal{H}$  is defined by a 1-form

$$(6) \quad \omega = dz + \omega_0 + \omega_1 z + \omega_2 z^2$$

where  $\omega_0, \omega_1, \omega_2$  are rational 1-forms on  $X$ . The integrability of  $\mathcal{H}$ ,  $\omega \wedge d\omega = 0$ , is equivalent to the equations

$$(7) \quad \begin{cases} d\omega_0 = \omega_0 \wedge \omega_1 \\ d\omega_1 = 2\omega_0 \wedge \omega_2 \\ d\omega_2 = \omega_1 \wedge \omega_2 \end{cases}$$

Here,  $\mathcal{F}$  is defined by  $\omega_0$ . A foliation  $\mathcal{F}$  on  $\mathbb{P}^n$  is transversely projective if, and only if, there exist rational 1-forms  $\omega_0, \omega_1, \omega_2$  on  $X$  satisfying (7) where  $\omega_0$  defines the foliation  $\mathcal{F}$ .

If there exists a transversely projective structure  $(P, \mathcal{H}, \sigma)$  for  $\mathcal{F}$  in which  $\omega_2 = 0$  (or equivalently there exists a section  $\tilde{\sigma} : X \dashrightarrow P$  invariant by  $\mathcal{H}$ ) then we say that  $\mathcal{F}$  is a transversely affine foliation.



If there exists a transversely projective structure  $(P, \mathcal{H}, \sigma)$  for  $\mathcal{F}$  in which  $\omega_2 = \omega_1 = 0$  then  $\mathcal{F}$  is a transversely Euclidean foliation. In particular, a foliation  $\mathcal{F}$  is transversely Euclidean if and only if  $\mathcal{F}$  is given by a closed rational 1-form.

Let  $f : \mathbb{P}^1 \rightarrow X$  be a morphism generically transverse to a foliation  $\mathcal{F}$ . If  $(P, \mathcal{H}, \sigma)$  is a transverse projective structure and the image of  $f$  is not contained in the indeterminacy locus of  $\sigma : X \dashrightarrow P$  then the naturally defined triple  $(f^*P, f^*\mathcal{H}, f^*\sigma)$  gives a projective structure on  $\mathbb{P}^1$ . In Section 6 we are going to explore this fact in order to define a foliation  $\mathcal{P}$  on  $\text{Mor}(\mathbb{P}^1, X)$  with leaves corresponding to morphisms inducing the same projective structure on  $\mathbb{P}^1$ . There we will need the following lemma which is due to Guy Casale.

**Lemma 5.1.** *Let  $F : Y \dashrightarrow X$  be a dominant rational map between projective manifolds,  $\mathcal{F}$  be a codimension one foliation on  $X$ , and  $\mathcal{G} = F^*\mathcal{F}$  be the foliation induced by  $\mathcal{F}$  on  $Y$ . Then the following assertions hold true.*

- (1) *The foliation  $\mathcal{G}$  is transversely projective if and only if  $\mathcal{F}$  is transversely projective.*
- (2) *The foliation  $\mathcal{G}$  is transversely affine if and only if  $\mathcal{F}$  is transversely affine.*

We note that  $\mathcal{G}$  might be transversely Euclidean while  $\mathcal{F}$  being only transversely affine. The simplest examples being linear foliations on abelian surfaces which are invariant by multiplication by  $-1$  while the defining 1-forms are not. Thus on the quotient, we have foliations which are transversely affine but not transversely Euclidean.

*Proof.* If  $\mathcal{F}$  is transversely projective (resp. affine or Euclidean) then  $\mathcal{G} = F^*\mathcal{F}$  is transversely projective (resp. affine or Euclidean) since such a structure  $(P, \mathcal{H}, \sigma)$  for  $\mathcal{F}$  pulls back to a similar structure  $(F^*P, F^*\mathcal{H}, F^*\sigma)$  for  $\mathcal{G}$ .

Suppose now that  $\mathcal{G}$  is transversely projective (resp. affine). Restrict  $\mathcal{G}$  and its projective structure to a sufficiently general submanifold having the same dimension as  $X$ . This reduces the problem to case where  $F$  is a generically finite rational map and we can apply [16, Lemme 2.1, Lemme 3.1] to conclude.  $\square$

## 6. DEFORMATION OF FREE MORPHISMS ALONG FOLIATIONS

**6.1. Deformation of free morphisms.** Let  $X$  be a projective manifold of dimension  $n$ . The morphisms from  $\mathbb{P}^1$  to  $X$  are parametrized by a locally Noetherian scheme  $\text{Mor}(\mathbb{P}^1, X)$ , [35, Theorem I.1.10]. The Zariski tangent space of  $\text{Mor}(\mathbb{P}^1, X)$  at a given morphism  $f : \mathbb{P}^1 \rightarrow X$  is canonically identified with  $H^0(\mathbb{P}^1, f^*TX)$  [22, Proposition 2.4] [35, Theorem I.2.16]. To understand this, suppose  $\text{Mor}(\mathbb{P}^1, X)$  is smooth at a point  $[f]$ , and let  $\gamma : (\mathbb{C}, 0) \rightarrow \text{Mor}(\mathbb{P}^1, X)$  be a germ of holomorphic curve in  $\text{Mor}(\mathbb{P}^1, X)$  such that  $\gamma(0) = [f]$ . If we fix  $x \in \mathbb{P}^1$  and compute  $\gamma'(0)(x)$  we obtain a vector at  $T_{f(x)}X \simeq (f^*TX)_x$ . Thus  $\gamma'(0) \in H^0(\mathbb{P}^1, f^*TX)$ .

For an arbitrary morphism  $f$ , the local structure of  $\text{Mor}(\mathbb{P}^1, X)$  at a neighborhood of  $[f]$  can be rather nasty, but if  $h^1(X, f^*TX) = 0$  then  $\text{Mor}(\mathbb{P}^1, X)$  is smooth and has dimension  $h^0(\mathbb{P}^1, f^*TX)$  at a neighborhood of  $[f]$ , see [35, Theorem I.2.16] or [22, Theorem 2.6].

If  $[f] \in \text{Mor}(\mathbb{P}^1, X)$  then Birkhoff-Grothendieck's Theorem implies that  $f^*TX$  splits as a sum of line bundles  $\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$  with  $a_1 \geq a_2 \geq \cdots \geq a_n$ . The morphism  $f$  is called free when  $a_n \geq 0$ . Notice that  $h^1(\mathbb{P}^1, f^*TX) = 0$  when  $f$  is a free morphism. Therefore  $\text{Mor}(\mathbb{P}^1, X)$  is smooth of dimension  $h^0(\mathbb{P}^1, f^*TX) = n + \sum_{i=1}^n a_i$  at a neighborhood of  $[f]$ .

The scheme  $\text{Mor}(\mathbb{P}^1, X)$  comes together with an evaluation map

$$\begin{aligned} F : \mathbb{P}^1 \times \text{Mor}(\mathbb{P}^1, X) &\longrightarrow X \\ (x, [f]) &\longmapsto f(x). \end{aligned}$$

Let  $f$  be a free morphism and  $M = M_f$  be the irreducible component of  $\text{Mor}(\mathbb{P}^1, X)$  containing  $[f]$ . The evaluation map  $F$  has maximal rank at any point of a neighborhood of  $\mathbb{P}^1 \times \{[f]\}$  in  $\mathbb{P}^1 \times M$  [35, Corollary II.3.5.4]. Indeed, it has maximal rank at a neighborhood of any point of the  $\text{Aut}(\mathbb{P}^1)$ -orbit of  $[f]$  under the natural action of  $\text{Aut}(\mathbb{P}^1)$  on  $\text{Mor}(\mathbb{P}^1, X)$  defined by right composition.

**6.2. Tangential foliation on the space of morphisms.** Let  $\mathcal{D}$  be a distribution on  $X$ , i.e.  $\mathcal{D}$  is determined by a coherent subsheaf of  $TX$  with torsion free cokernel as a foliation but without the involutiveness. We will say that a germ of deformation  $f_t : \mathbb{P}^1 \rightarrow X$ ,  $t \in (\mathbb{C}, 0)$  of a free morphism  $f = f_0 : \mathbb{P}^1 \rightarrow X$  is tangent to  $\mathcal{D}$  if the curves  $f_t(x) : (\mathbb{C}, 0) \rightarrow X$  are tangent to the distribution  $\mathcal{D}$  for every  $x$  in  $\mathbb{P}^1$ . These deformations correspond to germs of curves on  $M$  tangent to a distribution  $\mathcal{D}_{\text{tang}} = \mathcal{D}_{\text{tang}}(f)$  on  $M$  which we will call the tangential distribution of  $\mathcal{D}$ .

The construction of  $\mathcal{D}_{\text{tang}}$  is rather simple. From the identification  $T_{[f]}M \simeq H^0(\mathbb{P}^1, f^*TX)$  it follows that  $TM \simeq \pi_* F^*TX$ , where  $\pi : \mathbb{P}^1 \times M \rightarrow M$  is the natural projection. Thus the inclusion  $T\mathcal{D} \hookrightarrow TX$  gives rise to a morphism  $\pi_* F^*T\mathcal{D} \rightarrow TM$ . If  $\mathcal{I}$  denotes its image then we define  $\mathcal{D}_{\text{tang}}$  as the distribution on  $M$  determined by the saturation of  $\mathcal{I}$  inside  $TM$ , i.e.,  $T\mathcal{D}_{\text{tang}}$  is the smallest subsheaf of  $TM$  containing  $\mathcal{I}$  and with torsion free cokernel.

**Proposition 6.1.** *For a general free morphism  $g : \mathbb{P}^1 \rightarrow X$ ,  $[g] \in M$ , any germ of deformation  $g_t : \mathbb{P}^1 \rightarrow X$ ,  $t \in (\mathbb{C}, 0)$ , of  $g = g_0$  tangent to  $\mathcal{D}$  gives rise to a germ of curve  $[g_t] : (\mathbb{C}, 0) \rightarrow M$  tangent to  $\mathcal{D}_{\text{tang}}$ .*

*Proof.* Since  $g$  is general we may assume that its image is disjoint from the singular set of  $\mathcal{D}$ . By semi-continuity, there exists a non-empty open subset  $U \subset M$  where the sheaf  $\pi_* F^*T\mathcal{D}$  is locally free. If a germ of deformation  $g_t : \mathbb{P}^1 \rightarrow X$ ,  $t \in (\mathbb{C}, 0)$ , of a morphism  $g = g_0 \in U$  is tangent to  $\mathcal{D}$  then the corresponding germ of curve  $g_t : (\mathbb{C}, 0) \rightarrow M$  is clearly tangent to  $\mathcal{D}_{\text{tang}}$ .  $\square$

**Proposition 6.2.** *If the distribution  $\mathcal{D}$  is closed under Lie brackets then the same holds true for  $\mathcal{D}_{\text{tang}}$ .*

*Proof.* It suffices to verify at a neighborhood of a general morphism  $[g] \in M$ . We can assume for instance that the image of  $g$  is disjoint from the singular set of  $\mathcal{D}$ , thus  $\mathcal{D}$  foliates a neighborhood of  $g(\mathbb{P}^1)$ . If  $\xi_1, \xi_2$  are germs of sections of  $T\mathcal{D}_{\text{tang}}$  at  $[g]$  then the orbits of the corresponding vector field give rise to deformations of morphisms  $\phi_1, \phi_2 : (\mathbb{C}, 0) \times \mathbb{P}^1 \rightarrow X$  with  $\phi_i(0, x)g(x)$  and  $\partial_t \phi_i(t, x) \in T_{\phi_i(t, x)}\mathcal{D}$  for  $i = 1, 2$ . The involutiveness of  $\mathcal{D}$  implies that  $[\partial_t \phi_1(0, x), \partial_t \phi_2(0, x)] \in T_x \mathcal{D}$  for  $x \in \mathbb{P}^1$ . The involutiveness of  $\mathcal{D}_{\text{tang}}$  follows.  $\square$

**Remark 6.3.** The idea of studying a foliation through the induced foliations on the space of morphisms is not new and can be traced backed to [45]. While there it is explored to prove the uniruledness of the ambient manifold, here we will explore the uniruledness of the ambient manifold (through the existence of free rational curves) to unravel the structure of the original foliation.

**6.3. Interpretation.** Throughout this section we will suppose  $f : \mathbb{P}^1 \rightarrow X$  is free and generically transverse to a codimension one foliation  $\mathcal{F}$ , i.e., the general morphism in  $M \subset \text{Mor}(\mathbb{P}^1, X)$  is not tangent to  $\mathcal{F}$ . In what follows  $\mathcal{G} = F^*\mathcal{F}$  denotes the pull-back of  $\mathcal{F}$  under the evaluation morphism  $F : \mathbb{P}^1 \times M \rightarrow X$  and  $\mathcal{F}_{\text{tang}}$  is the tangential foliation induced on  $M$  as defined above.

If  $f^*N\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}(\nu + 2)$  then  $\mathcal{G}$  is defined by a 1-form

$$(8) \quad \Omega = \left( \sum_{i=0}^{\nu} a_i z^i \right) dz + \sum_{i=0}^{\nu+2} z^i \omega_i =: p(z)dz + \omega(z)$$

where  $p \in \mathbb{C}(M)[z]$  is a polynomial in  $z$  of degree at most  $\nu$  whose coefficients  $a_i$  are rational functions on  $M$  and  $\omega_0, \omega_1, \dots, \omega_{\nu+2}$  are rational 1-forms on  $M$  with  $\omega_{\nu+2} \neq 0$ . Notice that the zeros of  $p$  correspond to the tangencies between  $\mathcal{G}$  and the fibration  $\mathbb{P}^1 \times M \rightarrow M$  or, equivalently, to the tangencies between the corresponding morphisms and the foliation  $\mathcal{F}$ . Due to the natural action of  $\text{PGL}(2, \mathbb{C})$  on  $\mathbb{P}^1 \times M$  and the above geometrical interpretation, both leading coefficients  $a_\nu$  and  $a_0$  are non zero. After dividing  $\Omega$  by  $a_\nu$ , we can assume  $p(z)$  monic (and  $a_0 \neq 0$ ). This will be assumed from now on.

When  $\nu = 0$ , then  $\mathcal{G}$  is transversal to the fibration  $\mathbb{P}^1 \times M \rightarrow M$ , thus a Riccati foliation. When  $X = \mathbb{P}^n$  and  $M$  is the variety of linear morphisms (the lines of  $\mathbb{P}^n$  equipped with a projective coordinate), then  $\nu$  is precisely the degree of  $\mathcal{F}$ .

The foliation  $\mathcal{F}_{\text{tang}}$  is the foliation on  $M$  defined by the 1-forms  $\omega_0, \omega_1, \dots, \omega_{\nu+2}$ , as the restriction of  $\mathcal{G}$  to  $\mathbb{P}^1 \times L$ , where  $L$  is a leaf of  $\mathcal{F}_{\text{tang}}$ , is by construction determined by the 1-form  $dz$ .

**6.4. Algebraic closure of the leaves of the tangential foliation.** According to [9], there is a foliation  $\overline{\mathcal{F}}_{\text{tang}} \supset \mathcal{F}_{\text{tang}}$  with algebraic leaves such that for a general  $x \in X$ , the leaf of  $\overline{\mathcal{F}}_{\text{tang}}$  passing through  $x$  is algebraic and coincides with the Zariski closure of the corresponding leaf of  $\mathcal{F}_{\text{tang}}$ . We shall discuss the codimension of  $\mathcal{F}_{\text{tang}}$  in  $\overline{\mathcal{F}}_{\text{tang}}$ , and see how it controls the transverse geometry of  $\mathcal{G}$ .

**Theorem 6.4.** *The foliation  $\mathcal{F}_{\text{tang}}$  has codimension at most three in  $\overline{\mathcal{F}}_{\text{tang}}$ . Moreover,*

- (1) *If  $\mathcal{F}_{\text{tang}} = \overline{\mathcal{F}}_{\text{tang}}$  then  $\mathcal{G}$  is a pull-back from a manifold of dimension  $\dim M + 1 - \dim \mathcal{F}_{\text{tang}}$ .*
- (2) *If  $\text{codim}[\overline{\mathcal{F}}_{\text{tang}} : \mathcal{F}_{\text{tang}}] = 1$  or  $2$  then  $\mathcal{G}$  is the pull-back of a transversely affine Riccati foliation on  $\mathbb{P}^1 \times M$  by a rational map of the form  $(z, x) \mapsto (\alpha(x, z), x)$ . In particular,  $\mathcal{G}$  is transversely affine.*
- (3) *If  $\text{codim}[\overline{\mathcal{F}}_{\text{tang}} : \mathcal{F}_{\text{tang}}] = 3$  then  $\mathcal{G}$  is the pull-back of a Riccati foliation on  $\mathbb{P}^1 \times M$  by a rational map of the form  $(z, x) \mapsto (\alpha(x, z), x)$ . In particular,  $\mathcal{G}$  is transversely projective.*

The Frobenius integrability condition  $\Omega \wedge d\Omega = 0$  for  $\mathcal{G}$  can be reduced to

$$d\Omega = \Omega \wedge L_v \Omega \quad \text{with} \quad v = \frac{1}{p(z)} \frac{\partial}{\partial z}$$

where  $L_v \Omega := d(i_v \Omega) + i_v(d\Omega)$  is the Lie derivative of  $\Omega$  with respect to the vector field  $v$ . Indeed,  $i_v \Omega = \Omega(v) = 1$  and we get

$$0 = i_v(\Omega \wedge d\Omega) = (i_v \Omega) \cdot d\Omega - \Omega \wedge (i_v \Omega) = d\Omega - \Omega \wedge L_v \Omega;$$

the converse is obvious. If we denote by  $d_M$  the exterior differential of  $M$  and by  $'$  the derivative with respect to  $z$ , then integrability condition can be succinctly written as

$$(9) \quad p(z)d_M\omega(z) = \omega(z) \wedge \omega'(z) + d_M p(z) \wedge \omega(z).$$

Let  $K \subset \mathbb{C}(M)$  be the field of first integrals for  $\mathcal{F}_{tang}$ .

**Lemma 6.5.** *The coefficients of the polynomial  $p(z) = z^\nu + \sum_{i=0}^{\nu-1} a_i z^i$  belong to the field  $K$ .*

*Proof.* It suffices to prove that  $L_v a_i = 0$  for  $i = 0, \dots, \nu - 1$  for any rational vector field  $v$  on  $M$  tangent to  $\mathcal{F}_{tang}$ . Such a vector field, characterized by  $\omega_i(v) = 0$  for  $i = 0, \dots, \nu + 2$ , can be lifted as an horizontal vector field  $\tilde{v}$  on  $\mathbb{P}^1 \times M$  tangent to  $\mathcal{G}$ . Of course we have  $\Omega(\tilde{v}) = 0$ . The integrability condition implies

$$i_{\tilde{v}}(\Omega \wedge d\Omega) = L_{\tilde{v}}\Omega \wedge \Omega = 0.$$

Thus  $L_{\tilde{v}}\Omega$  is zero, or it defines the foliation  $\mathcal{G}$ . But

$$L_{\tilde{v}}\Omega = L_v p(z)dz + L_v \omega(z) = \left( \sum_{i=0}^{\nu} (L_v a_i) z^i \right) dz + \sum_{i=0}^{\nu+2} z^i L_v \omega_i$$

cannot be a non zero multiple of  $\Omega$  since  $L_v p$  has degree  $< \nu$  (recall that  $p$  is monic). Therefore, we are in the former case and  $L_v a_i = 0$  for  $i = 0, \dots, \nu - 1$  as wanted.  $\square$

**Lemma 6.6.** *For  $k = 0, \dots, \nu + 2$ , we have*

$$d\omega_k = \sum_{\substack{0 \leq i < j \leq k+1 \\ i+j \leq k+1}} \lambda_{ij} \omega_i \wedge \omega_j + \sum_{\substack{0 \leq i < j \leq k \\ i+j \leq k}} \mu_{ij} da_i \wedge \omega_j$$

for suitable  $\lambda_{ij}, \mu_{ij} \in K$ . Moreover, if  $i + j = k + 1$  then  $\lambda_{ij}$  is non-zero.

*Proof.* Equation (9) implies

$$(10) \quad \begin{cases} a_0 d\omega_0 & = \omega_0 \wedge \omega_1 + da_0 \wedge \omega_0, \\ a_0 d\omega_1 + a_1 d\omega_0 & = 2\omega_0 \wedge \omega_2 + da_0 \wedge \omega_1 + da_1 \wedge \omega_0, \\ a_0 d\omega_2 + a_1 d\omega_1 + a_2 d\omega_0 & = 3\omega_0 \wedge \omega_3 + \omega_1 \wedge \omega_2 \\ & \quad + da_0 \wedge \omega_2 + da_1 \wedge \omega_1 + da_2 \wedge \omega_0, \\ \vdots & = \vdots \\ \sum_{i=0}^k a_i d\omega_{k-i} & = \sum_{i=0}^k (k-i+1) \omega_i \wedge \omega_{k-i+1} \\ & \quad + \sum_{i=0}^k da_i \wedge \omega_{k-i} \end{cases}$$

The lemma follows inductively.  $\square$

**Lemma 6.7.** *Let  $n$  be the codimension of  $\mathcal{F}_{tang}$  in  $M$ . Then  $\Theta := \omega_0 \wedge \dots \wedge \omega_{n-1}$  is a non trivial closed  $n$ -form defining  $\mathcal{F}_{tang}$ . Moreover, for  $k = n, \dots, \nu + 2$  we can write  $\omega_k = \sum_{i=0}^{n-1} f_i \omega_i$  with  $f_i \in K$ .*

*Proof.* Let  $n$  be the largest integer such that  $\Theta := \omega_0 \wedge \dots \wedge \omega_{n-1} \neq 0$ . We have to prove that  $\Theta \wedge \omega_i = 0$  for any  $i$ . From the natural action of  $\mathrm{PGL}(2, \mathbb{C})$  on  $\mathbb{P}^1 \times M$ , we have a 1-parameter transformation group  $\Phi_t$  inducing  $z \mapsto z + t$  on the  $\mathbb{P}^1$ -coordinate and say  $\phi_t : M \rightarrow M$  on the basis. This action preserves  $\mathcal{G}$  as well

as  $\Omega$  since the  $dz$  coefficient of  $\Phi_t^*\Omega$  is still monic. Therefore, for any  $t \in \mathbb{C}$ , the identity below holds true:

$$\Phi_t^*\Omega = \left( \sum_{i=0}^{\nu} a_i \circ \phi_t(z+t)^i \right) dz + \sum_{i=0}^{\nu+2} (z+t)^i \phi_t^* \omega_i = \Omega.$$

From Taylor formula we get

$$\sum_{i=0}^{\nu+2} (z+t)^i \phi_t^* \omega_i = \phi_t^* \omega(z+t) = \phi_t^* \left( \sum_{i=0}^{\nu+2} z^i \omega^{(i)}(t) \right)$$

(where  $\omega^{(i)}$  denotes the  $i^{\text{th}}$  derivative of  $\omega(z)$  with respect to  $z$ ) and we deduce that  $\omega(t) \wedge \omega^{(1)} \wedge \dots \wedge \omega^{(n)}(t) = (\phi_t^{-1})^*(\omega_0 \wedge \dots \wedge \omega_n) = 0$  for all  $t$ . Expanding this equality in  $t$ -powers shows inductively that  $\omega_0 \wedge \dots \wedge \omega_{n-1} \wedge \omega_k = 0$  (i.e.  $\omega_k = \sum_{i=0}^{n-1} f_i \omega_i$ ) for all  $k = n, \dots, \nu+2$ .

Now, we claim that  $d\omega_i \wedge \hat{\Theta}_i = 0$  for any  $i = 0, \dots, n-1$  where

$$\hat{\Theta}_i := \omega_0 \wedge \dots \wedge \widehat{\omega_i} \wedge \dots \wedge \omega_{n-1} = 0$$

(here,  $\widehat{\omega_i}$  means that this term is omitted in the wedge product). In order to see this, go back to (10) and notice that

$$a_0 d\omega_k = (k+1)\omega_0 \wedge \omega_{k+1} + da_0 \wedge \omega_k + \dots$$

where  $(\dots)$  is a sum of wedge products of 1-forms involving at least one  $\omega_l$ , with  $l < k$  (here we assume  $n > 1$ ). Therefore

$$d\omega_i \wedge \hat{\Theta}_i = da_0 \wedge \Theta$$

which is zero by Lemma 6.5. On the other hand, if  $n = 1$ , then  $\omega_0 \wedge \omega_1 = 0$  and  $a_0 d\omega_0 = da_0 \wedge \omega_0 = 0$  (same reason). We promptly deduce that  $d\Theta = \sum_{i=0}^{n-1} d\omega_i \wedge \hat{\Theta}_i = 0$ .

Now, let  $\omega_k$  be any other coefficient of  $\Omega$ ; one can write  $\omega_k = \sum_{i=0}^{n-1} f_i \omega_i$  for some (unique)  $f_i \in \mathbb{C}(M)$ . Then

$$d\omega_k \wedge \omega_0 \wedge \dots \wedge \widehat{\omega_k} \wedge \dots \wedge \omega_{n-1} = \pm df_k \wedge \Theta$$

must be zero by the very same argument, and  $f_k$  actually belongs to  $K$ .  $\square$

The field  $K$  may now be defined as follows:

$$K = \{f \in \mathbb{C}(M) ; df \wedge \Theta = 0\}.$$

It is an integrally closed field and, according to Siegel's Theorem, there exists a dominant rational map  $\Phi : M \dashrightarrow N$  onto a projective variety  $N$  such that  $K = \Phi^*\mathbb{C}(N)$ . The dimension of  $N$  coincides with the transcendence degree  $p := [K : \mathbb{C}]$ . The general fibers of  $\Phi$  are the (algebraic closures of) the leaves of  $\overline{\mathcal{F}}_{tang}$ . One can choose  $f_1, \dots, f_p \in K$  such that the closed  $p$ -form  $df_1 \wedge \dots \wedge df_p \neq 0$  defines  $\overline{\mathcal{F}}_{tang}$ , or equivalently  $K$  is the integral closure of  $\mathbb{C}(f_1, \dots, f_p)$ . Let now  $q = \text{codim}[\overline{\mathcal{F}}_{tang} : \mathcal{F}_{tang}]$ ,  $n = p + q$ . By very similar arguments, one can prove the alternate  $K$ -relative version of the previous lemma.

**Lemma 6.8.** *The foliation  $\mathcal{F}_{tang}$  is defined by the non trivial closed  $n$ -form*

$$\Theta := \omega_0 \wedge \dots \wedge \omega_{q-1} \wedge df_1 \wedge \dots \wedge df_p, \quad q + p = n$$

*while the algebraic closure  $\overline{\mathcal{F}}_{tang}$  is defined by  $df_1 \wedge \dots \wedge df_p$ . Moreover, every other  $\omega_k$  in (8) can be written as  $\omega_k = \sum_{i=0}^{q-1} b_i \omega_i + \sum_{i=q}^{n-1} b_i df_{i-q+1}$  with all  $b_i \in K$ .*

**Lemma 6.9.** *The codimension  $q = \text{codim}[\overline{\mathcal{F}}_{\text{tang}} : \mathcal{F}_{\text{tang}}]$  is at most 3.*

*Proof.* Following Lemma 6.8, we can write

$$\omega_q = \sum_{i=0}^{q-1} b_i \omega_i + \sum_{i=q}^{n-1} b_i df_{i-q+1}.$$

According to equation (10), we have

$$a_0 d\omega_{q-1} + \cdots + a_{q-1} d\omega_0 = q\omega_0 \wedge \omega_q + (q-2)\omega_1 \wedge \omega_{q-1} + (q-4)\omega_2 \wedge \omega_{q-2} + \cdots$$

After plugging the expression of  $\omega_q$  into this equation and differentiating, we get an equality between 3-forms. They both decompose uniquely in terms of  $\omega_i \wedge \omega_j \wedge \omega_k$ ,  $\omega_i \wedge \omega_j \wedge df_k$  and  $\omega_i \wedge df_j \wedge df_k$ , where the subscripts of the 1-forms  $\omega_i$  range over  $0, \dots, q-1$  and the subscripts of the functions  $f_i$  range over  $1, \dots, p$ . The term  $\omega_0 \wedge \omega_2 \wedge \omega_{q-1}$  does not occur on the left hand side, and Lemma 6.6 implies that only the terms

$$0 = \cdots + (q-2)d\omega_1 \wedge \omega_{q-1} + \cdots - (q-4)\omega_2 \wedge d\omega_{q-2} + \cdots$$

contribute on the right hand side. Thus we arrive at the identity

$$0 = q(q-3)\omega_0 \wedge \omega_2 \wedge \omega_{q-1}$$

which contradicts the integrability conditions if  $q > 3$ .  $\square$

**6.5. Proof of Theorem 6.4.** Suppose first that  $\text{codim}[\overline{\mathcal{F}}_{\text{tang}} : \mathcal{F}_{\text{tang}}] = 3$ . Let us denote by  $V$  the  $K$ -vector space  $K\langle \omega_0, \omega_1, \omega_2, df_1, \dots, df_p \rangle$ , where  $f_1, \dots, f_p$  are the rational functions given by Lemma 6.8. The  $K$ -subspace of  $V$  generated by  $df_1, \dots, df_p$  will be denoted by  $W$ ; and  $\underline{V}$  will denote the quotient of  $V$  by  $W$ . By assumption,  $\underline{V}$  has dimension 3. We note that both  $V$  and  $W$  are “closed under exterior differential” in the sense that  $dV = V \wedge V$  and  $dW = W \wedge W$ ; it follows that the exterior derivative is well-defined on the quotient  $\underline{V}$ , which is itself closed as well. Therefore by the classification of three dimensional Lie algebras there exists a basis  $\vartheta_0, \vartheta_1, \vartheta_2$  for  $\underline{V}$  satisfying the structure equations of  $\mathfrak{sl}(2, \mathbb{C})$  in  $\underline{V}$ :

$$\begin{cases} d\vartheta_0 &= \vartheta_0 \wedge \vartheta_1, \\ d\vartheta_1 &= 2\vartheta_0 \wedge \vartheta_2, \\ d\vartheta_2 &= \vartheta_1 \wedge \vartheta_2. \end{cases} \quad \text{modulo } W.$$

Notice that the other possibilities are excluded since  $\vartheta_0 \wedge \vartheta_1 \wedge \vartheta_2$ , or the wedge product of any other basis, must be different from zero. We claim that we can choose  $\vartheta_i \in V$  satisfying the same equations in  $V$  instead of  $\underline{V}$ . To prove this claim, let us start by expanding  $\frac{\Omega}{p(z)}$  in a power series

$$dz + \frac{\omega_0 + z\omega_1 + z^2\omega_2 + \cdots + z^{n+2}\omega_{n+2}}{a_0 + a_1z + a_2z^2 + \cdots + z^n} = dz + \eta_0 + z\eta_1 + z^2\eta_2 + z^3\eta_3 + \cdots$$

Then the classes of  $\eta_0, \eta_1, \eta_2$  in  $\underline{V}$  clearly defines a new basis for it. The integrability conditions in  $V$  then become

$$\begin{aligned} d\eta_0 &= \eta_0 \wedge \eta_1, \\ d\eta_1 &= 2\eta_0 \wedge \eta_2, \\ d\eta_2 &= 3\eta_0 \wedge \eta_3 + \eta_1 \wedge \eta_2, \\ d\eta_3 &= 4\eta_0 \wedge \eta_4 + 2\eta_1 \wedge \eta_3, \\ &\vdots \end{aligned} \tag{11}$$

If  $\eta_0 \wedge \eta_3 = 0$  then we are done. Otherwise Lemma 6.8 allows us to write

$$\eta_3 = a\eta_0 + b\eta_1 + c\eta_2 + \sum_{i=1}^p g_i df_i$$

with  $a, b, c, g_i \in K$ . If we take the exterior derivative of the third line of (11) we get

$$0 = d(d\eta_2) = 6c \cdot \eta_0 \wedge \eta_1 \wedge \eta_2 \mod W \wedge V \wedge V$$

which implies  $c = 0$ . If  $f \in K$  and we replace  $z$  by  $z + fz^3$  in the differential form  $dz + \eta_0 + z\eta_1 + z^2\eta_2 + z^3\eta_3 + \dots$  then we modify the sequence  $\eta_2, \eta_3, \eta_4, \dots$  without modifying the integrability equations (11). To wit,  $\eta_2$  is changed to  $\eta_2 - 3f \cdot \eta_0$  and  $\eta_3$  to  $\eta_3 - 3f\eta_1 + df$ . We use this operation to obtain a new  $\eta_3$  which takes the form  $\eta_3 = a\eta_0 + \sum_{i=1}^p g_i df_i \in W$ . Finally, if we take the wedge product of  $\eta_0$  with the exterior derivative the fourth line of (11) then we get

$$\eta_0 \wedge d\eta_3 = \eta_0 \wedge \eta_1 \wedge \left( \sum_{i=1}^p g_i df_i \right).$$

If we see  $V$  as the direct sum  $\underline{V} \oplus W$  then the lefthand side of the equality above lies in  $\underline{V} \wedge \bigwedge^2 W$ , while the righthand side lies in  $\bigwedge^2 \underline{V} \wedge W$ . It follows that  $\eta_3 = a\eta_0$ , and the claim is proved.

Now, we can write

$$\tilde{\Omega} = \frac{\Omega}{p(z)} = dz + a(z)\eta_0 + b(z)\eta_1 + c(z)\eta_2 + \sum_{i=1}^p g_i(z)df_i$$

where  $a, b, c, g_i \in K(z)$ . If we expand  $\tilde{\Omega} \wedge d\tilde{\Omega}$  then we get

$$(a + a'b - ab')\eta_0 \wedge \eta_1 \wedge dz + (2b + b'c - bc')\eta_0 \wedge \eta_2 \wedge dz + (c + b'c - bc')\eta_1 \wedge \eta_2 \wedge dz + \beta$$

where  $\beta$  is a 3-form with monomials involving at most one of the 1-forms  $\eta_0, \eta_1, \eta_2$ . Therefore

$$0 = (a + a'b - ab') = (2b + b'c - bc') = (c + b'c - bc').$$

The first equality can be rewritten as  $\frac{b'}{b} - \frac{a'}{a} = \frac{1}{b}$ . After setting  $\alpha(z) = \frac{b}{a}$ , we get  $\frac{\alpha'}{\alpha} = \frac{1}{b}$  and thus

$$a = \frac{1}{\alpha'} \quad \text{and} \quad b = \frac{\alpha}{\alpha'}.$$

A combination of the 3 equations yields  $b^2 = ac$  and we thus get  $c = \frac{\alpha^2}{\alpha'}$ . Now, we can rewrite

$$\alpha' \tilde{\Omega} = d\alpha + \eta_0 + \alpha\eta_1 + \alpha^2\eta_2 + \sum_{i=1}^p h_i(z)df_i$$

for suitable  $h_i \in K(z)$ . Examining again the integrability condition, we obtain

$$0 = \eta_0 \wedge \eta_1 \wedge \left( \sum_{i=1}^p h_i(z)df_i \right) + \beta$$

where  $\beta$  is, as before, a 3-form with monomials involving at most one of the 1-forms  $\eta_0, \eta_1, \eta_2$ . Thus the functions  $h_i$  vanish identically and

$$\alpha' \tilde{\Omega} = d\alpha + \eta_0 + \alpha\eta_1 + \alpha^2\eta_2.$$

This is sufficient to conclude that  $\mathcal{G}$  is defined by the pull-back of  $dz + \eta_0 + z\eta_1 + z^2\eta_2$  under the rational map  $(z, x) \mapsto (\alpha(z), x)$ .

Let us now turn to the case  $\text{codim}[\overline{\mathcal{F}}_{tang} : \mathcal{F}_{tang}] = 2$ . As the arguments are very similar to the previous case, we will just sketch them. In this case there are  $\eta_0, \eta_1$  satisfying

$$\begin{aligned} d\eta_0 &= \eta_0 \wedge \eta_1, \\ d\eta_1 &= 2\eta_0 \wedge \eta_2, \end{aligned}$$

with  $\eta_2 = a\eta_1 + \sum_{i=1}^p g_i df_i$  with  $a, g_i \in K$ . After killing the coefficient  $a$  by adding to  $\eta_1$  a convenient  $K$ -multiple of  $\eta_0$ , the closedness of  $d\eta_1$  gives  $g_i = 0$ , and thus  $d\eta_1 = 0$ . If we write  $\tilde{\Omega} = p(z)^{-1}\Omega = dz + a(z)\eta_0 + b(z)\eta_1 + \sum_{i=1}^p h_i(z)df_i$  then the integrability of  $\tilde{\Omega}$  implies again  $\frac{b'}{b} - \frac{a'}{a} = \frac{1}{b}$  so that we can set  $a = \frac{1}{\alpha'}$  and  $b = \frac{\alpha}{\alpha'}$  as before. Rewriting the integrability condition for  $\alpha'\tilde{\Omega}$  gives  $h_i = 0$ . Therefore

$$\alpha'\tilde{\Omega} = d\alpha + \eta_0 + \alpha\eta_1$$

and  $\mathcal{G}$  is defined by the pull-back of  $dz + \eta_0 + z\eta_1$  under the rational map  $(z, x) \mapsto (\alpha(z), x)$ .

When  $\text{codim}[\overline{\mathcal{F}}_{tang} : \mathcal{F}_{tang}] = 1$ , integrability conditions write

$$d\eta_0 = \eta_0 \wedge \eta_1$$

with  $\eta_1 = \sum_{i=1}^p g_i df_i$ . Closedness of  $d\eta_0$  gives  $d\eta_1 = 0$ . We can therefore proceed analogously to the previous case, proving that  $\mathcal{G}$  is defined by the pull-back of  $dz + \eta_0 + z\eta_1$  under the rational map  $(z, x) \mapsto (\alpha(z), x)$ .

Finally, when  $\overline{\mathcal{F}}_{tang} = \mathcal{F}_{tang}$ , it suffices to notice that the leaves of  $\mathcal{F}_{tang}$  lift to leaves of  $\mathcal{G}$ , and define a subfoliation of  $\mathcal{G}$  by algebraic leaves of the same dimension as  $\mathcal{F}_{tang}$ .  $\square$

**Remark 6.10.** A weaker version of Theorem 6.4 can be deduced in an easier way as a corollary of [19, Theorem 1.1]. Indeed, write

$$\Omega_0 := \frac{\Omega}{p(z)} = dz + g_1(z)\omega_1 + \cdots + g_n(z)\omega_n$$

where  $\Theta = \omega_1 \wedge \cdots \wedge \omega_n \neq 0$ . We can assume that  $W := \text{Wronskian}(g_1, \dots, g_n) \neq 0$ , otherwise we could write  $\Omega_0$  with less summands. Then a Godbillon-Vey sequence for  $\mathcal{G}$  (see [19]) is given by

$$\begin{aligned} \Omega_1 &= g'_1\omega_1 + \cdots + g'_n\omega_n, \\ \Omega_2 &= g''_1\omega_1 + \cdots + g''_n\omega_n, \\ \Omega_3 &= g'''_1\omega_1 + \cdots + g'''_n\omega_n, \\ &\vdots \end{aligned}$$

Since  $\Omega_0 \wedge \Omega_1 \wedge \cdots \wedge \Omega_n = W \cdot dz \wedge \Theta$  and  $\Omega_0 \wedge \Omega_1 \wedge \cdots \wedge \Omega_n \wedge \Omega_{n+1} \neq 0$  we can apply [19, Theorem 1.1] to deduce that  $\mathcal{G}$  is either transversely projective, or pull-back as in the statement of Theorem 6.4. Actually, our proof above is somehow dual to that one of [19]: there we dealt with vector fields instead of differential forms. Notice that here we obtain a stronger result, since we obtain that  $\mathcal{G}$  is a pull-back of a Riccati equation in the transversely projective case. In particular, this excludes the possibility of  $\mathcal{F}$  being a transversely hyperbolic foliation like the *Hilbert modular foliations*, see [59, Theorem III.2.6].



**6.6. Variation of projective structure.** In this subsection we will assume that  $\mathcal{F}$  admits a transversely projective structure. When the leaves of  $\mathcal{F}_{tang}$  are non algebraic, we will compare them with the levels of a rational function on  $M$  naturally determined by the transverse projective structure. This will allow us to show that at neighborhood of a general free rational curve the original foliation  $\mathcal{F}$  is defined by a closed meromorphic 1-form.

Let  $\mathcal{F}$  be a transversely projective on a projective manifold  $X$ . Recall that attached to  $\mathcal{F}$  we have the datum  $(P, \mathcal{H}, \sigma)$  where  $p : P \rightarrow X$  is  $\mathbb{P}^1$ -bundle over  $X$ ,  $\mathcal{H}$  is a Riccati foliation on  $P$ , and  $\sigma : X \dashrightarrow P$  is a rational section. After making birational gauge transformations we can (and will) assume that  $P$  is the trivial  $\mathbb{P}^1$ -bundle over  $X$ . We will denote by  $\Delta$  the polar divisor of  $\mathcal{H}$ , i.e.,  $\Delta$  is the divisor on  $X$  such that  $p^*\Delta$  is the tangency divisor between  $\mathcal{H}$  and the one-dimensional foliation defined by the fibers of  $p$ . In any local trivialization of the bundle  $P$ ,  $\Delta$  is the polar divisor of the Riccati 1-form (6) defining  $\mathcal{H}$ .

If  $f : \mathbb{P}^1 \rightarrow X$  is a free morphism which is generically transverse to  $\mathcal{F}$  then  $\mathcal{F}$ , or rather its first integrals, define a (singular) projective structure on  $\mathbb{P}^1$ . If we consider the irreducible component of  $\text{Mor}(\mathbb{P}^1, X)$  containing  $[f]$ , we can associate to a general element of  $M$  a rational quadratic differential  $\phi(x)dx^2$  on  $\mathbb{P}^1$  using equation (5) from Section 5. As the order of poles of  $\phi(x)dx^2$  is uniformly bounded as a function of  $[f] \in M$ , this explicit formula defines a rational map from  $M$  to the space of rational quadratic differentials. The foliation on  $M$  determined by this rational map will be denoted by  $\mathcal{S}$ . The foliation  $\mathcal{F}_{tang}$  is clearly tangent to the foliation  $\mathcal{S}$ , i.e. the general leaf of  $\mathcal{F}_{tang}$  is contained in a leaf of  $\mathcal{S}$ .

**Theorem 6.11.** *If  $\dim \mathcal{S} > \dim \mathcal{F}_{tang}$  then for a general  $[f] \in M$  the Riccati foliation  $f^*\mathcal{H}$  defined on  $f^*P$  is defined by a closed rational 1-form. Moreover, if  $f$  is an embedding then there exists a neighborhood  $U \subset X$  of  $f(\mathbb{P}^1)$  in the metric topology where  $\mathcal{F}$  is defined by a closed meromorphic 1-form.*

*Proof.* Let  $T = (\mathbb{C}, 0)$  and fix a germ of curve  $\gamma : T \rightarrow M$  with image contained in a leaf of  $\mathcal{S}$  and transverse to the leaf of  $\mathcal{F}_{tang}$  through  $\gamma(0) = [f]$ . Let  $\Sigma = \mathbb{P}^1 \times T$  and  $F_\gamma = F \circ \gamma : \Sigma \rightarrow X$  be the composition of the evaluation morphism with  $\gamma$ .

Pulling back  $\mathcal{H}$  using  $F_\gamma$  we obtain a Riccati foliation  $\tilde{\mathcal{H}}$  in  $\Sigma \times \mathbb{P}^1$  which is defined by

$$dz + \omega_0 z^2 + \omega_1 z + \omega_2$$

where  $\omega_0, \omega_1, \omega_2$  are 1-forms in the variables  $(x, t) \in \Sigma = \mathbb{P}^1 \times T$ ; and the section  $\sigma$  defined by  $z = \infty$ .

We can apply a bimeromorphic transformation of the form  $z \mapsto \alpha(x, t)z + \beta(x, t)$  in order to write

$$\Omega = dz + \left( z^2 + \frac{\phi(x, t)}{2} \right) dx + (a(x, t)z^2 + b(x, t)z + c(x, t))dt,$$

as a 1-form defining  $\tilde{\mathcal{H}}$ . Since we are in a leaf of  $\mathcal{S}$  we indeed get that  $\partial_t \phi(x, t) = 0$ , i.e.  $\phi(x, t) = \phi(x)$  does not depend on  $t$ . Hence we can write

$$\Omega = dz + (dx + a(x, t)dt)z^2 + (b(x, t)dt)z + \left( \frac{\phi(x)}{2} dx + c(x, t)dt \right).$$

The restriction of  $\tilde{\mathcal{H}}$  to  $\{z = \infty\}$  is defined by the 1-form  $dx + a(x, t)dt$ . It can be identified with the foliation  $\mathcal{G}_\Sigma = F_\gamma^* \mathcal{F}$ . Since  $\gamma : T \rightarrow M$  is not contained in a leaf of  $\mathcal{F}_{tang}$ , it follows that  $\mathcal{G}_\Sigma$  does not coincide with the foliation defined by  $dx$ .

Thus the function  $a(x, t)$  is not identically zero. Since  $[f] \in M$  is general, we can assume that  $a(x, 0)$  is not identically zero.

Notice that  $\Omega = i_v i_w dt \wedge dx \wedge dz$  where

$$v = -\partial_t + (a(x, t)z^2 + b(x, t)z + c(x, t))\partial_z \quad \text{and} \quad w = -\partial_x + (z^2 + \phi(x))\partial_z.$$

Due to the particular form of these vector fields, the involutiveness of  $T\tilde{\mathcal{H}}$  is equivalent to  $[v, w] = 0$ . As  $w$  does not depend on  $t$ , it commutes with  $\partial_t$ . Therefore it also commutes with  $(a(x, 0)z^2 + b(x, 0)z + c(x, 0))\partial_z$ , and this vector field is an infinitesimal automorphism of the Riccati foliation  $\mathcal{R}$  on  $\mathbb{P}^1 \times \mathbb{P}^1$  obtained by restricting  $\tilde{\mathcal{H}}$  to  $\mathbb{P}^1 \times \{[f]\} \times \mathbb{P}^1$ . Consequently the 1-form

$$\tilde{\Omega} = \frac{dz + (z^2 + \phi(x))dx}{a(x, 0)z^2 + b(x, 0)z + c(x, 0)}$$

is a closed rational 1-form defining  $\mathcal{R}$ .

If  $f$  is an embedding then  $C = f(\mathbb{P}^1) \subset X$  is smooth curve. Since  $f$  is free we can assume that  $C$  intersects the polar divisor of  $\mathcal{H}$  generically and that it is disjoint from the indeterminacy locus of  $\sigma$ , see [35, Proposition II.3.7]. Thus [39, Lemma 4.1] implies that for every point  $p \in C$  the germ of  $\mathcal{H}$  at  $\pi^{-1}(p)$  admits a local product structure. That is, there is a local system of coordinates  $(x_1, \dots, x_n, y)$  on  $(X, p) \times \mathbb{P}^1$  where the natural projection to  $X$  is the morphism forgetting  $y$ ,  $C$  is defined by  $x_2 = \dots = x_n = 0$  and the foliation  $\mathcal{H}$  is defined by  $\Omega_p = a(x_1, y)dx_1 + b(x_1, y)dy$ . The restriction of  $\Omega_p$  to  $\pi^{-1}(C)$  is a multiple of  $\tilde{\Omega}$  and therefore we can write  $\tilde{\Omega} = h(x_1, y)\Omega_p$ . It follows that  $h(x_1, y)\Omega_p$  is an extension of  $\tilde{\Omega}$  at the neighborhood of  $p$  and it still defines  $\mathcal{H}$ . Notice that such extension is unique. Therefore, the 1-form  $\tilde{\Omega}$  extends as a closed meromorphic 1-form defining  $\mathcal{H}$  at a neighborhood of  $\pi^{-1}(C)$ . It suffices to pull-back  $\tilde{\Omega}$  using  $\sigma$  to obtain a closed meromorphic 1-form defining  $\mathcal{F}$  at a neighborhood of  $C$ .  $\square$

**6.7. Graphic neighborhood.** The deformations of a morphism  $f : \mathbb{P}^1 \rightarrow X$  along a foliation  $\mathcal{F}$  can be interpreted as deformations of another morphism  $\Gamma_f : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times X$  – the graph of  $f$  – which are contained in an analytic subvariety  $Y$  of  $\mathbb{P}^1 \times X$ . To keep things simple suppose that the image of  $f$  is disjoint from the singular set of  $\mathcal{F}$ . On each fiber of  $\mathbb{P}^1 \times X \rightarrow \mathbb{P}^1$  put an unaltered copy of the foliation  $\mathcal{F}$  to form a codimension 2 foliation  $\hat{\mathcal{F}}$  on  $\mathbb{P}^1 \times X$ . Let  $\Delta$  be the graph of  $f$  in  $\mathbb{P}^1 \times X$ , and  $U$  an arbitrarily small tubular neighborhood of  $\Delta$  in the metric topology. If we saturate  $\Delta$  by the leaves of  $\hat{\mathcal{F}}|_U$ , we obtain a smooth analytic subvariety of  $U$  of dimension  $\dim \mathcal{F} + 1$  which we will call  $Y$ . The nice thing about  $Y$  is that the normal bundle of  $\Delta$  in  $Y$  coincides with the tangent sheaf of  $\mathcal{F}$ . More precisely,

$$\Gamma_f^* N_Y \Delta = f^* T\mathcal{F}.$$

Bogomolov-McQuillan explored this fact in [8] to establish the algebraicity of the leaves under the assumption that the tangent sheaf of a foliation is ample when restricted to a curve.

Let  $\pi_1 : Y \rightarrow \mathbb{P}^1$  and  $\pi_2 : Y \rightarrow X$  be the natural projections. By construction,  $\pi_2^* \mathcal{F}$  coincides with the foliation on  $Y$  defined by the fibers of  $\pi_1$ .

Notice that a sufficiently small deformations of  $\Delta$  in  $Y$  are graphs of morphisms of  $\mathbb{P}^1$  which deform  $f$  along  $\mathcal{F}$ , and reciprocally a deformation of  $f : \mathbb{P}^1 \rightarrow X$  along  $\mathcal{F}$  gives rise to a family of curves on  $Y$ . This alternative interpretation for deformation of morphisms along  $\mathcal{F}$  will be essential in the proof of our next result.

Although we will apply it only for codimension one foliations, it does hold without any assumptions on the codimension of  $\mathcal{F}$ .

**Proposition 6.12.** *Let  $[g] \in M \subset \text{Mor}(\mathbb{P}^1, X)$  be a general element with  $g(\mathbb{P}^1)$  not everywhere tangent to  $\mathcal{F}$ , and let  $k$  be the number of summands of  $g^*T\mathcal{F}$  having strictly positive degree. If  $x \in X$  is a general point then there exists a quasi-projective variety  $V_x$  of dimension at least  $k$  passing through  $x$  and contained in the leaf of  $\mathcal{F}$  through  $x$ .*

*Proof.* Let  $Y \subset \mathbb{P}^1 \times X$  be a graphic neighborhood of  $g$ , and  $\Delta$  be the graph of  $g$  in  $Y$ . If  $L$  is the leaf of  $\mathcal{F}_{\text{tang}}$  through  $[g]$  and  $\varphi : \mathbb{P}^1 \times L \rightarrow Y \subset \mathbb{P}^1 \times X$  is the morphism defined by  $\varphi(z, [h]) = (z, h(z))$  then the differential

$$d\varphi : T\mathbb{P}^1 \times TL \longrightarrow T\mathbb{P}^1 \oplus \varphi^*(TX)$$

of  $\varphi$  at  $(z, [h])$  is given by

$$d\varphi(z, [h]) = \text{id}_{T\mathbb{P}^1} \oplus (dh(z) + \phi(z, [h]))$$

where  $\phi(z, [h]) : H^0(\mathbb{P}^1, h^*T\mathcal{F}) \rightarrow h^*TX \otimes \mathcal{O}_X/\mathfrak{m}_z\mathcal{O}_X$  is the evaluation morphism, and  $dh$  is the differential of  $h$ , see [35, page 114]. If  $z_0 \in \mathbb{P}^1$  and  $[g]$  are general enough then the kernel of  $d\varphi(z_0, [g])$  has dimension equal to  $h^0(\mathbb{P}^1, g^*T\mathcal{F} \otimes \mathfrak{m}_{z_0})$ , where  $\mathfrak{m}_{z_0} \subset \mathcal{O}_{\mathbb{P}^1, z_0}$ . Therefore  $A = \varphi^{-1}(\varphi(z_0, [g])) \subset \{z_0\} \times L$  also has dimension  $h^0(\mathbb{P}^1, g^*T\mathcal{F} \otimes \mathfrak{m}_{z_0})$ . The morphisms parametrized by the projection of  $A \subset \mathbb{P}^1 \times L$  to  $L$  have image contained in  $Y$  and containing  $p = (z_0, g(z_0))$ . Thus we have an analytic family of morphisms  $T$  (the projection of  $A$ ) contained in  $M \subset \text{Mor}(\mathbb{P}^1, X)$ , all of them with graph contained in  $Y$  and containing  $p$ . The description of  $d\varphi$  given above implies that for a general  $y \in \mathbb{P}^1$ , the image under  $\varphi$  of  $\{y\} \times T$  has dimension at least  $k$ , the number of non-negative summands of  $g^*T\mathcal{F} \otimes \mathfrak{m}_{z_0} \cong g^*T\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$ .

Let  $\overline{T}$  be the Zariski closure of  $T$  in  $M$ . Every point of  $\overline{T}$  corresponds to a morphism with graph containing  $p$ . Let denote the maximal ideal of the local ring of  $\mathbb{P}^1$  at  $z_0$ . For  $k \geq 0$ , let

$$\psi_k(h) = (j_k \text{id}, j_k h) : \text{Spec} \left( \frac{\mathcal{O}_{\mathbb{P}^1, z_0}}{\mathfrak{m}_{z_0}^{k+1}} \right) \rightarrow \mathbb{P}^1 \times X$$

be the map induced by  $\text{id}_{\mathbb{P}^1} \times h$  at the corresponding  $k$ -th infinitesimal neighborhood. If  $h$  corresponds to a point in  $T$  then  $\psi_k(h)$  factors through the inclusion  $Y \hookrightarrow X$  for any  $k \geq 0$ . In other words, if  $\mathcal{I}_{Y,p} \subset \widehat{\mathcal{O}}_{\mathbb{P}^1 \times X, p}$  is the ideal of formal functions of  $X$  at  $p$  generated by formal functions vanishing along  $Y$  then  $\psi_k^*(\mathcal{I}_{Y,p}) = 0$  for any  $h$  corresponding to a point in  $T$ . By (Zariski) continuity, the same holds true for any  $h$  corresponding to a general point in  $\overline{T}$ . Therefore if  $h : \mathbb{P}^1 \rightarrow X$  is a morphism corresponding to a point of  $\overline{T}$  then the formal germ of  $\text{id}_{\mathbb{P}^1} \times h$  at  $z_0$  will map the formal germ of  $\mathbb{P}^1$  at  $z_0$  to  $Y$ . Therefore  $\overline{T}$  is a  $s$ -dimensional (algebraic) deformation of  $g$  along  $\mathcal{F}$  sending  $z_0$  to  $g(z_0)$ , i.e.  $\overline{T}$  is tangent to  $\mathcal{F}_{\text{tang}}$ .

If  $F : \mathbb{P}^1 \times \overline{T} \rightarrow X$  is the restriction to  $\overline{T}$  of the evaluation morphism then for a general  $y \in \mathbb{P}^1$ ,  $F(\{y\} \times \overline{T})$  will be a constructible set of dimension at least  $\dim F(\{y\} \times T) \geq k$  contained in the leaf of  $\mathcal{F}$  through  $g(y)$ . The proposition follows.  $\square$

We do not know how to control the geometry of the quasi-projective varieties  $V_x$  constructed by the previous proposition. It is conceivable that a variation of Mori's bend-and-break argument would allow us to construct a  $k$ -dimensional rationally connected subvariety tangent to  $\mathcal{F}$  through a general point  $x \in X$ . So far we can

only prove the existence of one rational curve tangent to  $\mathcal{F}$  through a general point of  $X$ .

**Proposition 6.13.** *Let  $M \subset \text{Mor}(\mathbb{P}^1, X)$  be a irreducible component containing free morphisms and  $[g] \in M$  be a general element. Suppose  $g^*T\mathcal{F}$  has at least one summand having strictly positive degree. If  $x \in X$  is a general point then there exists rational curve through  $x$  and contained in the leaf of  $\mathcal{F}$  through  $x$ .*

*Proof.* The proof is similar to the one of [46, Lemma 5.2, Lecture I]. If  $g(\mathbb{P}^1)$  is tangent to  $\mathcal{F}$  then there is nothing to prove. Otherwise, according to Proposition 6.12, the existence of a positive summand in the decomposition of  $g^*T\mathcal{F}$  implies that we can algebraically deform  $g$  along  $\mathcal{F}$  in such a way that a general point  $z_0 \in \mathbb{P}^1$  is mapped to  $g(z_0) = x$  along the deformation. More precisely, there exists a smooth quasiprojective curve  $C^0 \subset M$  contained in a leaf of  $\mathcal{F}_{\text{tang}}$  and such that every  $[h] \in C^0$  maps  $z_0$  to  $x$ .

Let  $C$  be a smooth projective curve containing  $C^0$  as an open subset. The evaluation morphism  $F : \mathbb{P}^1 \times C^0 \rightarrow X$  extends to a rational map  $F : \mathbb{P}^1 \times C \dashrightarrow X$ . Generically  $F$  must have rank two, as otherwise the deformation would have to move points along the image  $g(\mathbb{P}^1)$  of one of its member and this is only possible if  $g(\mathbb{P}^1)$  is tangent to  $\mathcal{F}$ . Notice also that  $F^*\mathcal{F}$  is nothing but the foliation on  $\mathbb{P}^1 \times C$  defined by the projection  $\mathbb{P}^1 \times C \rightarrow \mathbb{P}^1$ .

Since the curve  $C_0 = \{p_0\} \times C$  has self-intersection zero in  $\mathbb{P}^1 \times C$  and  $F$  has image of dimension two, there must exist an indeterminacy point of  $F$  on  $C_0$ . By resolving the indeterminacies of  $F$  we obtain a surface  $S$  together with a morphism  $G : S \rightarrow X$  fitting into the diagram

$$\begin{array}{ccc}
 S & & \\
 \downarrow \pi & \searrow G & \\
 \mathbb{P}^1 \times C & \xrightarrow{\quad F \quad} & X \\
 \uparrow & \nearrow F & \\
 \mathbb{P}^1 \times C^0 & & 
 \end{array}$$

where  $\pi : S \rightarrow \mathbb{P}^1 \times C$  is a birational morphism. Moreover, there exists a curve  $E \subset S$  contracted by  $\pi$  into a point of  $C_0$  whose image under  $G$  is a rational curve on  $X$  passing through  $x$ . Since the foliation  $F^*\mathcal{F}$  is a smooth foliation on  $\mathbb{P}^1 \times C$ , every exceptional divisor of  $\pi$  is also invariant by  $(F \circ \pi)^*\mathcal{F} = G^*\mathcal{F}$ . Therefore  $G(E)$  is the sought rational curve tangent to  $\mathcal{F}$  passing through  $x$ .  $\square$

**6.8. Proof of Theorem 5.** If the general leaf of  $\mathcal{F}_{\text{tang}}$  is not algebraic then Theorem 6.4 and Lemma 5.1 imply  $\mathcal{F}$  is transversely projective. If instead every leaf of  $\mathcal{F}_{\text{tang}}$  is algebraic then we can conclude applying the next proposition.

**Proposition 6.14.** *Let  $X$  be a uniruled projective manifold,  $M \subset \text{Mor}(\mathbb{P}^1, X)$  be an irreducible component containing free morphisms, and  $\mathcal{F}$  be a codimension one foliation on  $X$ . If all the leaves of the foliation  $\mathcal{F}_{\text{tang}}$  defined on  $M$  are algebraic then the foliation  $\mathcal{F}$  is the pull-back by a rational map of a foliation  $\mathcal{G}$  on a projective manifold of dimension smaller than or equal to  $n - \delta_0 + \delta_{-1}$ , where  $\delta_0 = h^0(\mathbb{P}^1, f^*T\mathcal{F})$ , and  $\delta_{-1} = h^0(\mathbb{P}^1, f^*T\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^1}(-1))$ . Moreover, if  $\delta_{-1} > 0$  then  $\mathcal{F}$  is uniruled.*

*Proof.* Let  $F : \mathbb{P}^1 \times M \rightarrow X$  be the evaluation morphism, and  $\mathcal{H}_{tang}$  be the foliation defined by intersection of  $F^*\mathcal{F}$  and the pull-back of  $\mathcal{F}_{tang}$  under the projection  $\mathbb{P}^1 \times M \rightarrow M$ . Notice that  $\mathcal{H}_{tang}$  has the same dimension as  $\mathcal{F}_{tang}$  and, if the leaves of  $\mathcal{F}_{tang}$  are all algebraic then the same holds true for the leaves of  $\mathcal{H}_{tang}$ . Thus under our hypothesis  $\mathcal{H}_{tang}$  provides a family of algebraic subvarieties tangent to  $F^*\mathcal{F}$ .

The dimension of  $\mathcal{H}_{tang}$  is exactly  $\delta_0$ , and if we restrict the evaluation map to a general leaf of  $\mathcal{H}_{tang}$  it follows that its rank is exactly  $\delta_0 - \delta_{-1}$ . Therefore through a general point of  $X$  there exists an algebraic subvariety of dimension  $\delta_0 - \delta_{-1}$  tangent to  $\mathcal{F}$ . Lemma 2.4 implies that  $\mathcal{F}$  is a pull-back from a variety having dimension at most  $n - \delta_0 + \delta_{-1}$ . To produce rational curves tangent to  $\mathcal{F}$  when  $\delta_{-1} > 0$  it suffices to apply Proposition 6.13.  $\square$

Suppose now that  $X$  is rationally connected and  $f : \mathbb{P}^1 \rightarrow X$  is an embedding with ample normal bundle. Theorem 6.11 implies that  $\mathcal{F}$  is defined by a closed meromorphic 1-form  $\omega$  at a neighborhood of  $f(\mathbb{P}^1)$ . We can apply [31, Theorem 6.7] to extend  $\omega$  to an algebraic (perhaps multi-valued) 1-form on all of  $X$ . It follows that  $\mathcal{F}$  is transversely affine. Indeed the algebraicity of  $\omega$  implies the existence of a projective manifold  $Y$  together with a generically finite morphism  $p : Y \rightarrow X$  such that  $p^*\mathcal{F}$  is defined by a closed rational 1-form.  $\square$

**6.9. Proof of Corollary 6.** Theorem 5 implies that (a)  $\mathcal{F}$  is a pull-back of a foliation  $\mathcal{G}$  on a projective manifold of dimension at most  $n - \delta_0 + \delta_{-1}$ ; or (b)  $\mathcal{F}$  is transversely affine. If we are in case (a) with  $\delta_{-1} > 0$  then the foliation  $\mathcal{F}$  is uniruled. If we are still in case (a) but now with  $\delta_{-1} = 0$  then  $\delta_0 = n - 1$ , and  $\mathcal{F}$  is the pull-back of a foliation by points on a curve, i.e., the leaves of  $\mathcal{F}$  are all algebraic. In this case  $\mathcal{F}$  is defined by a closed rational 1-form. If we are in case (b) then there exists a pair of rational 1-forms  $\omega, \eta$  such that  $\omega$  defines  $\mathcal{F}$  and  $\eta$  is a closed rational 1-form satisfying  $d\omega = \eta \wedge \omega$ . Let  $D$  be the divisor of poles of  $\eta$ . The obstruction to  $\mathcal{F}$  be defined by a closed rational 1-form is given by the monodromy representation

$$\begin{aligned} \pi_1(X \setminus |D|) &\longrightarrow \mathbb{C}^* \\ \gamma &\longmapsto \exp \left( \int_{\gamma} \eta \right). \end{aligned}$$

If this representation is trivial  $\mathcal{F}$  is defined by a closed rational 1-form. Suppose this is not the case.

According to [36] there exists a rational curve  $C \subset X$  with ample normal bundle such that  $\pi_1(C \setminus C \cap |D|)$  surjects to  $\pi_1(X \setminus |D|)$ . We can repeat the argument used in the proof of Theorem 5 to produce another transversely affine structure (now defined by rational 1-forms  $\omega'$  and  $\eta'$ ) but now  $\eta'$  has trivial monodromy at a neighborhood of  $C$ . If we write  $\omega = h\omega'$  then  $d\omega = (\eta' - \frac{dh}{h}) \wedge \omega$ . Thus  $(\eta - \eta' + \frac{dh}{h}) \wedge \omega = 0$ , and  $\eta - \eta' + \frac{dh}{h}$  is the sought non-zero closed rational 1-form defining  $\mathcal{F}$ .  $\square$

**6.10. Proof of Corollary 7.** If the general leaf of  $\mathcal{F}_{tang}$  is not algebraic then we can use the same argument as in the proof of Corollary 6, replacing the use of [36] by the standard Lefschetz's Theorem, to conclude that  $\mathcal{F}$  is defined by a closed rational 1-form.

If all the leaves of  $\mathcal{F}_{\text{tang}}$  are algebraic then we want to control  $\delta_0 - \delta_{-1}$  for a general linear immersion  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^n$ , as  $\mathcal{F}$  will be a pull-back from a projective manifold of dimension at most  $n - \delta_0 + \delta_{-1}$  according to Theorem 5. Write  $f^*T\mathcal{F}$  as

$$\mathcal{O}_{\mathbb{P}^1}(1)^r \oplus \mathcal{O}_{\mathbb{P}^1}^s \oplus \bigoplus_{i=1}^{(n-1)-r-s} \mathcal{O}_{\mathbb{P}^1}(b_i)$$

where  $b_i < 0$ . Therefore  $\delta_0 - \delta_{-1} = r + s$  and  $\deg(f^*T\mathcal{F}) = (n-1) - d = r + \sum b_i \leq r - ((n-1) - (r+s)) = 2r + s - (n-1)$ . Thus

$$\delta_0 - \delta_{-1} = r + s \geq \frac{1}{2}(2r + s) \geq n - 1 - \frac{d}{2},$$

which implies that  $n - \delta_0 + \delta_{-1} \leq d/2 + 1$  as wanted.  $\square$

## 7. FOLIATIONS WITH NUMERICALLY TRIVIAL CANONICAL BUNDLE

In this section we will conclude the proof of our main result.

**Theorem 7.1** (Theorem 1 of the Introduction). *Let  $\mathcal{F}$  be a codimension one foliation with numerically trivial canonical bundle on a projective manifold  $X$ . Then at least one of following assertions hold true.*

- (a) *The foliation  $\mathcal{F}$  is defined by a closed meromorphic 1-form with coefficients in a torsion line bundle and without divisorial components in its zero set.*
- (b) *All the leaves of  $\mathcal{F}$  are algebraic.*
- (c) *The foliation  $\mathcal{F}$  is uniruled.*

Moreover, if  $\mathcal{F}$  is not uniruled then  $K\mathcal{F}$  is a torsion line bundle.

Let us briefly recall what we already know. On the one hand, if  $\mathcal{F}$  does not satisfy none of the conclusions of Theorem 7.1 then Corollary 3.9 and Corollary 4.9 imply that the singular set of  $\mathcal{F}$  has at least one component of codimension two, and that the general point of every component of  $\text{sing}(\mathcal{F})$  the foliation admits a local holomorphic first integral. On the other hand, Theorem 5 implies that  $\mathcal{F}$  is transversely projective. We will use this information to proof the existence of a divisor satisfying the hypothesis of the lemma below.

### 7.1. Logarithmic division implies smoothness or uniruledness.

**Lemma 7.2.** *Let  $\mathcal{F}$  be a codimension one foliation on a projective manifold  $X$  defined by a twisted 1-form  $\omega \in H^0(X, \Omega_X^1 \otimes N\mathcal{F})$ . Suppose there exists a closed analytic subset  $R \subset X$  of codimension at least 3, a  $\mathbb{R}$ -divisor  $D = \sum \lambda_i H_i$  with  $\lambda_i > -1$  and  $H_i$  irreducible  $\mathcal{F}$ -invariant hypersurface such that for every  $x \in X \setminus R$  we can write locally*

$$(12) \quad \omega \wedge \left( \sum \lambda_i \frac{dh_i}{h_i} \right) = d\omega$$

where  $h_1, \dots, h_k$  are local equations for  $H_1, \dots, H_k$ . Then  $H^1(X, N^*\mathcal{F}) \neq 0$

*Proof.* One can assume for simplicity that  $R = \emptyset$ . Indeed, if (12) holds on  $X \setminus R$ , it holds also on the whole  $X$  by extension properties of  $H^1(X \setminus R, N^*\mathcal{F})$  through analytic subsets of codimension  $\geq 3$ . Let  $\mathcal{U} = \{U_\alpha\}$  be a sufficiently fine open covering of  $X$ . If  $\{g_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})\}$  is a cocycle representing  $N\mathcal{F}$  then, according

to our hypothesis, the collection of holomorphic 1-forms  $\{\theta_{\alpha\beta} \in \Omega_X^1(U_{\alpha\beta})\}$  defined by

$$\frac{dg_{\alpha\beta}}{g_{\alpha\beta}} - \left( \sum \lambda_i \frac{dh_{i,\beta}}{h_{i,\beta}} - \sum \lambda_i \frac{dh_{i,\alpha}}{h_{i,\alpha}} \right)$$

vanishes along the leaves of  $\mathcal{F}$ . Thus we have an induced class in  $H^1(X, N^*\mathcal{F})$ . If non-zero there is nothing else to prove. Otherwise, we deduce that  $N^*\mathcal{F}$  is numerically equivalent to  $D$ .

By assumption, there exists near every point  $x \in X$  a function  $\varphi$  expressed locally as  $\sum \varphi_i$  with  $\varphi_i$  plurisubharmonic and  $\frac{i}{2\pi} \partial \bar{\partial} \varphi_i = \lambda_i [H_i]$  as currents. Moreover, adding if necessary a pluriharmonic function, we get the following equalities

$$(13) \quad \omega \wedge \partial \varphi = d\omega \quad \text{and} \quad \partial \varphi = \sum \lambda_i \frac{dh_i}{h_i}.$$

Since  $N^*\mathcal{F}$  is numerically equivalent to  $D$ , we can interpret  $\varphi$  as a local weight of a (singular) metric on  $N^*\mathcal{F}$ . The expression

$$T = ie^\varphi \omega \wedge \bar{\omega}$$

gives rise to a closed positive  $(1,1)$ -current on  $U = X \setminus \text{sing}(\mathcal{F})$ . Indeed,  $e^\varphi$  is locally integrable on  $U$  since  $\lambda_i > -1$ , and  $T$  is closed thanks to (13). Beware that  $e^\varphi$  may fail to be integrable near some point of the singular locus. Nevertheless,  $T$  extends uniquely to a closed positive current on  $X$  denoted again  $T$  by a slight abuse of notations. Remark that  $T$  splits on  $U$  as  $-\eta \wedge \omega$  where  $\eta = ie^\varphi \bar{\omega}$  is a well defined  $(0,1)$   $\bar{\partial}$ -closed current with values in  $N\mathcal{F}^*$ .

Now, pick a point  $p \in \text{sing}(\mathcal{F})$  and denote by  $B_p$  an open ball centered at  $p$ . On  $B_p$ , there exists a plurisubharmonic potential  $\psi$  for  $T$ :

$$i\partial\bar{\partial}\psi = T.$$

From  $i\bar{\partial}(\partial\psi \wedge \omega) = 0 = -T \wedge \omega = 0$ , we can deduce the existence of a distribution  $\delta$ , defined at least on  $B_p \setminus \text{sing}(\mathcal{F})$  such that

$$\partial\psi = \delta\omega.$$

Consequently we have, up to a multiplicative constant, the identity

$$e^\varphi \bar{\omega} = \bar{\partial}\delta.$$

Therefore  $\eta$  is  $\bar{\partial}$ -exact on  $B_p \setminus \text{sing}(\mathcal{F})$ . By Mayer-Vietoris, the class  $\{\eta\}$  defined in  $H^1(U, N^*\mathcal{F})$  extends to a class in  $H^1(X, N^*\mathcal{F})$ . The positivity of  $T$  implies that this class is non-trivial and the lemma follows.  $\square$

**7.2. Transversely projective structure, Schwartz derivative, and invariant divisors.** Suppose  $\mathcal{F}$  is a codimension one foliation on a projective manifold  $X$  with numerically trivial canonical bundle. We will also assume that at the general point of every irreducible component of the singular set of  $\mathcal{F}$  having codimension two, the foliation is defined by

$$pxdy + qydx$$

with  $p, q$  relatively prime positive integers. Moreover, we can also assume that for at least one of the irreducible codimension two components of  $\text{sing}(\mathcal{F})$  the integers  $p$  and  $q$  are distinct thanks to Corollary 4.9. We will now make use of the transversely projective structure given by Theorem 5 to produce a divisor satisfying the hypothesis of Lemma 7.2. The degeneracy locus of such a structure is of the form  $\text{sing}(\mathcal{F}) \cup \Sigma$  where  $\Sigma$  is a finite union of  $\mathcal{F}$ -invariant hypersurfaces. Outside this set,

the foliation is defined by local submersions with values in  $\mathbb{P}^1$  and transition functions in  $\text{Aut}(\mathbb{P}^1)$ . We emphasize that the transverse structure gives distinguished first integrals for the foliation  $\mathcal{F}$  outside the degeneracy locus of the projective structure. We will denote the sheaf of such first integrals by  $\mathcal{I}$ .

Consider a regular point  $p \in X - \text{sing}(\mathcal{F})$  where the foliation is locally given by a submersion  $z$ ,  $z(p) = 0$ . We can select an open neighborhood  $U$  of  $p$  and a section  $f$  (possibly multi-valued) of  $\mathcal{I}$  (which depends only on the  $z$  variable) such that the Schwartz derivative of  $f$  with respect to  $z$ ,  $\{f, z\}$ , is a well defined meromorphic function on the whole open set  $U$ . Hence, we can expand the Schwartz derivative of  $f$  with respect to  $z$  as

$$\{f, z\} = \sum_{i \geq i_0} a_i z^i$$

with  $i_0 \in \mathbb{Z}$  and  $a_{i_0} \neq 0$ , unless  $\{f, z\}$  vanishes identically. The following facts can be easily verified.

- (1) The first integral  $f$  is a submersion if and only if  $i_0 \geq 0$ . In particular, if  $i_0 < 0$  then the local invariant hypersurface  $\{z = 0\}$  actually belongs to an algebraic hypersurface in  $\Sigma$ .
- (2) If  $i_0 \leq -1$  then it is independent of the choice of the local coordinate  $z$ . Consequently,  $i_0$  is constant along the irreducible hypersurfaces in  $\Sigma$ . If  $H$  is one of such hypersurfaces then we will denote by  $i_0(H)$  the value of  $i_0$  along it. Moreover, if  $i_0 \geq -2$  then the coefficient of  $\frac{1}{z^2}$  is independent of the coordinate and we define  $a(H) = a_{-2}$ .
- (3) the function  $f(z) - \log z$  is holomorphic if and only if  $i_0 = -2$  and  $a_{-2} = \frac{1}{2}$ .

We will say that  $H$  is an irregular singularity of the projective structure if and only if  $i_0(H) < -2$ . Otherwise, if  $i_0(H) \in \{-2, -1\}$  we will say that  $H$  is a regular singularity.

**7.2.1. Passing through corners.** Let  $\omega = pydx + qxdy$  be a germ of 1-form at the origin of  $\mathbb{C}^n$ , with  $p, q$  relatively prime positive integers. Suppose that the foliation  $\mathcal{F}$  induced by  $\omega$  is endowed with a projective structure. Let  $f$  be a multi-valued section of  $\mathcal{I}$  defined on the complement of  $\{xy = 0\}$ . Let  $r = \frac{q}{p}$ . Set  $i_x = i_0(\{x = 0\})$  and  $i_y = i_0(\{y = 0\})$ . On the transversals  $\{y = 1\}$  and  $\{x = 1\}$ , we get respectively

$$\{f, x\} = \sum_{i \geq i_x} a_i x^i \quad \text{and} \quad \{f, y\} = \sum_{i \geq i_y} b_i y^i.$$

These two restrictions are related by the so called Dulac's transform, a (multi-valued) holonomy transformation between the two transversals, which is explicitly given by  $x = h(y) = y^r$ .

The composition rule for the Schwartz derivative

$$\{f \circ h, z\} = \{f, h(z)\}h'(z)^2 + \{h, z\}$$

applied to  $x = h(y) = y^r$ , together with the fact that

$$\{h, y\} = \frac{1 - r^2}{y^2}$$

implies the next lemma.

**Lemma 7.3.** *The following assertions hold true.*



- (1) If the singularity on  $\{x = 0\}$  is irregular, i.e.  $i_x < -2$ , then  $i_y = r(i_x + 2) - 2$  and  $b_{i_y} = a_{i_x} r^2 (\neq 0)$ . Therefore,  $i_y < -2$  and the singularity on  $\{y = 0\}$  is also irregular.
- (2) If the singularity on  $\{x = 0\}$  is regular with  $i_x = -2$  then  $i_y \geq -2$  and  $b_{-2} = r^2(a_{i_x} - \frac{1}{2}) + \frac{1}{2}$ .
- (3) If  $i_x \geq -1$  and  $r \neq 1$  then  $i_y = -2$ .

It follows that the projective structure determines a *canonical* logarithmic 1-form  $\eta$  on  $\{xy = 0\}$  satisfying  $d\omega = \eta \wedge \omega$  as follows.

- (1) In case of irregular singularities:  $\eta = (-i_x - 3)\frac{dx}{x} + (-i_y - 3)\frac{dy}{y}$ .
- (2) In case of regular singularities:  $\eta = (|2a_{-2} - 1|^{\frac{1}{2}} - 1)\frac{dx}{x} + (|2b_{-2} - 1|^{\frac{1}{2}} - 1)\frac{dy}{y}$ .

Notice that in both cases the residues are real and strictly greater than  $-1$ .

**7.3. Proof of Theorem 7.1 (Theorem 1 of the Introduction).** Let  $\mathcal{F}$  be a transversely projective foliation which is of the form  $pdx + qdy$  ( $p, q$  relatively prime positive integers) at the general point of every codimension two irreducible component  $S$  of  $\text{sing}(\mathcal{F})$ . We will now construct a divisor  $D$  with support on  $\Sigma$  (the singular set of the transversely projective structure) satisfying the hypothesis of Lemma 7.2.

Write  $\Sigma = \Sigma_1 \cup \dots \cup \Sigma_\ell$  as the union of its connected components. Fix a connected component  $\Sigma_j$  and pick a point  $p \in \text{sing}(\mathcal{F}) \cup \Sigma_j$  in an irreducible component  $H_p \subset \Sigma_j$ .

Assume that  $i_0(H_p) < -2$ . Then every other irreducible component  $H$  of  $\Sigma_j$  must satisfy  $i(H) < -2$  according to Lemma 7.3. Then we set

$$D_j = \sum_{H \subset \Sigma_j} (-i(H) - 3)H.$$

Notice that  $D_j$  satisfies the hypothesis of Lemma 7.2 in a neighborhood of  $\Sigma_j$ .

Assume that  $i_0(H_p) = -2$  and  $a(H_p) = \frac{1}{2}$ . Lemma 7.3 implies the same holds true for every irreducible component  $H \subset \Sigma_j$ . Thus over a general point of  $\Sigma_j$  we get a logarithmic first integral (induced by the projective structure) which gives rise to a well defined local section  $\eta$  of  $d\mathcal{I}$ . Indeed, these local sections are logarithmic 1-form with poles on  $\Sigma_j$  which are unique up to a multiplicative constant. Using Mayer-Vietoris sequence, we deduce the existence of a global logarithmic form  $\eta_j$  on a neighborhood  $V$  of  $\Sigma_j$ . Moreover, one can choose its residues positive real numbers. In this case we set

$$D_j = \sum_{H \subset \Sigma_j} (\text{res}_H(\eta))H.$$

Assume that  $i_0(H_p) = -2$  et  $a(H_p) \neq \frac{1}{2}$ . In this case, using again Lemma 7.3, if we set

$$D_j = \sum_{H \subset \Sigma_j} (|2a(H) - 1|^{\frac{1}{2}} - 1)H$$

then  $D_j$  satisfies the hypothesis of Lemma 7.2 in a neighborhood of  $\Sigma_j$ .

If we sum up the divisors  $D_j$  for all the connected components of  $\Sigma$  we obtain a divisor  $D$  satisfying the hypothesis of Lemma 7.2, and consequently we get the non-vanishing of  $H^1(X, N^*\mathcal{F})$ . As we are assuming that  $\mathcal{F}$  is not uniruled, Proposition 4.5 implies  $\mathcal{F}$  is smooth. The explicit description given in [56] (see §1.1) implies that  $\mathcal{F}$  is defined by a closed holomorphic 1-form with coefficients in a torsion line

bundle. Therefore to conclude the proof of Theorem 7.1 it remains to show that the closed 1-forms given by Theorem 2 have coefficients in a torsion line bundle.

**7.3.1. Flat implies torsion.** Let  $\mathcal{F}$  be a non-uniruled foliation with  $c_1(K\mathcal{F}) = 0$ , given by closed rational form  $\omega$  without zeroes divisor and with coefficients in a flat line bundle  $L$ . Assume  $\text{sing}(\mathcal{F}) \neq \emptyset$  and write  $(\omega)_\infty = \sum \lambda_D D$  as a sum of irreducible divisors with positive integers coefficients.

Assume first that  $\lambda_D > 1$  for every  $D$ , i.e. the 1-form  $\omega$  is locally the differential of a meromorphic function. We can argue as in Theorem 3.8 to deduce that  $\mathcal{F}$  is uniruled, or has a rational first integral (with general leaf rationally connected), or every codimension two component of the singular set of  $\mathcal{F}$  admits a local holomorphic first integral of the type  $x^p y^q$ . In the latter case we can apply the arguments of this section to deduce that  $\mathcal{F}$  is uniruled or smooth. Therefore Theorem 7.1 is proved when  $\lambda_D > 1$  for every  $D$ .

Recall that rationally connected manifolds are simply-connected, and consequently  $L$  is trivial in these manifolds. Thus we have only to deal with  $X$  uniruled, with rational quotient  $R_X$  not reduced to a point, and  $\omega$  has at least one logarithmic pole, i.e. there is a divisor  $D$  in the support  $(\omega)_\infty$  with  $\lambda_D = 1$ .

Let us call  $\mathcal{F}_{rat}$  the codimension  $q = \dim R_X$  foliation with algebraic leaves induced by the rationally connected meromorphic fibration

$$R : X \dashrightarrow R_X.$$

We know [28] that  $R_X$  is not uniruled. Therefore [11] implies that  $\mathcal{F}_{rat}$  is given by an holomorphic  $q$ -form on  $X$  without zeroes in codimension 1 and with coefficients in a line bundle  $E$  such that  $E^*$  is pseudo-effective. The restriction of such  $q$ -form on the leaves defines a non trivial section  $\sigma$  of  $\Omega_{\mathcal{F}}^q \otimes E$ , where  $\Omega_{\mathcal{F}}^q$  denotes the  $q$ -th wedge power of the cotangent sheaf of  $\mathcal{F}$ . If  $\mathcal{F}$  is not uniruled then  $T\mathcal{F}$  is semi-stable with respect to any polarization of  $X$ . In this case the section  $\sigma$  has no zeroes in codimension 1. This property forces  $R|_D$  to be dominant over  $R_X$ . Notice, for later use, that the semi-stability of  $T\mathcal{F}$  and the pseudo-effectiveness of  $E^*$  implies that  $E$  is flat.

Let us denote by  $U \subset R_X$  the open Zariski subset such that the fibration  $R$  over  $U$  is a regular one. Let us pick a small open ball  $B$  on  $U$ . Over  $R^{-1}(B)$  our flat line bundle  $L$  is trivial since the fibers of  $R$  are rationally-connected. Therefore we can represent  $\omega$  in  $R^{-1}(U)$  by a meromorphic 1-form normalized in such a way that its residue along a branch of  $D$  is equal to 1. Since there are only finitely many choices involved, this enables us to conclude that  $L$  is torsion. The proof of Theorem 7.1 (Theorem 1 of the Introduction) is concluded.  $\square$

## 8. TOWARD A MORE PRECISE STRUCTURE THEOREM

In this section we refine the description of non uniruled codimension one foliations with numerically trivial canonical bundle when the ambient manifold  $X$  admits a semi-positive smooth  $(1,1)$ -form representing  $c_1(X)$ . For a structure theorem for this class of manifolds see [23].

**Theorem 8.1.** *Let  $X$  be a projective manifold carrying a smooth semi-positive closed  $(1,1)$ -form which represents  $c_1(X)$ . If  $\mathcal{F}$  is a codimension one foliation on  $X$  with  $c_1(K\mathcal{F}) = 0$  then at least one of the following assertions holds true.*

- (1) *The foliation  $\mathcal{F}$  is uniruled.*

- (2) *Up to a finite etale covering, the maximally rationally connected fibration of  $X$  is a locally trivial smooth fibration over a manifold  $B$  with zero canonical class and  $\mathcal{F}$  is obtained as a suspension of a representation of  $\pi_1(B)$  on the automorphism group of a codimension one foliation  $\mathcal{G}$  defined on the rationally connected fiber  $F$  and such that  $c_1(K\mathcal{G}) = 0$  on  $F$ .*
- (3) *Up to a finite etale covering,  $X$  is the product of a projective manifold  $B$  with trivial canonical class with a rationally connected projective manifold  $F$ . The foliation  $\mathcal{F}$  is defined by a closed rational 1-form of the form*

$$\omega = \alpha + \beta$$

*where  $\alpha$  is a closed rational form without divisorial zeroes defining a foliation  $\mathcal{G}$  on  $F$  and  $\beta$  is a closed holomorphic 1-form on  $B$ .*

We conjecture that the result above holds without the hypothesis on  $c_1(X)$ .

Notice that (3) implies (2) whenever  $\alpha$  admits a non trivial infinitesimal symmetry; i.e, there exists on  $F$  a holomorphic vector field such that  $\alpha(X) = 1$ . We do not know if this holds true in general.

**8.1. Smoothness of the rationally connected fibration.** The lemma below can be easily deduced from the structure theorem of [23]. We include a proof here for the sake of completeness.

**Lemma 8.2.** *Under the assumptions of Theorem 8.1, the rational quotient  $R_X$  is smooth with vanishing  $c_1$ . Moreover, the ambient manifold  $X$  is obtained as a suspension of a representation from  $\pi_1(R_X)$  to  $\text{Aut}(F)$  where  $F$  is a fiber of  $X \rightarrow R_X$ .*

*Proof.* The arguments laid down in §7.3.1 imply that the maximal rationally connected fibration is defined, as a foliation, by a holomorphic  $q$ -form  $\xi$  with values in a flat line bundle  $E$ , where  $q = \dim R_X$ .

Let us equip  $E$  with a metric  $h$  such that  $(E, h)$  is unitary flat. By assumption there exists a closed semi-positive  $(1, 1)$  form  $\alpha$  which represents  $c_1(X)$ . By Yau's theorem ([60]),  $\alpha$  turns out to be the Ricci form of a Kähler metric  $g$  on the whole manifold  $X$ .

Let us endow  $\Omega^q(X)$  with the induced metric  $g_q$  from the Kähler metric  $g$  on  $X$ , and the vector bundle  $F = \Omega^q(X) \otimes E$  with the metric  $g_q \otimes h$ . In this way,  $F$  becomes a Hermitian vector bundle. If we apply Hopf's maximum principle and the standard Bochner identity to the Laplacian of the function  $|\alpha|^2$  on  $X$ , the curvature condition implies that  $\alpha$  is parallel and nowhere vanishing. In particular, the rationally connected fibration is smooth and the subbundle  $S$  of  $TX$  defined by local holomorphic vector fields tangent to the fibers is indeed parallel. Thus we get, with respect to the metric  $g$ , an holomorphic splitting

$$TX = S \oplus S^\perp$$

with each member of the summand being an integrable subbundle. Furthermore, since the fibers of the rationally connected fibration are simply-connected, this fibration has no multiple fibers. The lemma follows.  $\square$

Using the description of Kähler manifolds with vanishing first Chern class ([4]), we may assume, up to a finite etale covering, that  $X = Y \times V$ , where  $V$  is Calabi-Yau ( $KV = \mathcal{O}_V$  and  $\pi_1(V) = 0$ ) and  $Y$  is a locally trivial rationally connected fibration over a complex torus  $T$  of dimension  $s$ . Moreover, the splitting  $S \oplus S^\perp$  of

$TX$  endows  $Y$  with a locally free  $\mathbb{C}^s$ -action transverse to the rationally connected fibration.

**Lemma 8.3.** *If  $H \subset TX$  is the relative tangent bundle of the fibration  $X \rightarrow Y$  with Calabi-Yau fibers then  $H \subset T_{\mathcal{F}}$ .*

*Proof.* Assume the inclusion does not hold. Then, on a general fiber  $\mathcal{F}$  induces a codimension one foliation  $\mathcal{F}_V$ . Adjunction implies that  $N^*\mathcal{F}|_V = KX|_V = KV$  is indeed trivial and therefore  $N^*\mathcal{F}_V$  is effective. This contradicts the vanishing of  $H^1(V, \mathbb{C})$ .  $\square$

It follows that  $\mathcal{F}$  is the pull-back of a foliation defined on  $Y$ . It is sufficient to restrict our attention to the case  $X = Y$  in order to prove Theorem 8.1.

**8.2. Automorphism group of  $X$ .** Keeping the notation of Section 7.3.1, consider the non-trivial effective divisor

$$\Delta = (\omega)_{\infty} = \sum \lambda_D D$$

Let us call  $\text{Aut}_0(X)$  the connected component of the identity of  $\text{Aut}(X)$ , the group of biholomorphisms of  $X$ . We have a natural action of  $\text{Aut}(X)$  on  $\text{Div}(X)$  compatible with the linear equivalence.

**Lemma 8.4.** *For every  $g \in \text{Aut}_0(X)$ ,  $g(\Delta)$  is linearly equivalent to  $\Delta$ .*

*Proof.* Following [6], there is a well defined a groups morphism

$$\text{Aut}_0(X) \rightarrow \text{Pic}^0(X)$$

sending  $g$  to the class of  $g(\Delta) - \Delta$ .

In our context, there exists by assumption a flat line bundle  $L$ , arising from a morphism  $\rho : \pi_1(X) \rightarrow \mathbb{C}^*$  and a holomorphic section  $\sigma$  of  $\det(TX) \otimes L$  with  $\Delta$  as zero divisor. This section lifts to a holomorphic section  $\tilde{\sigma}$  of the anticanonical bundle of the universal covering  $\tilde{X}$  such that  $h_*\tilde{\sigma} = \rho(h)\tilde{\sigma}$  for every covering transformation  $h$ . Now, let us consider the lift  $\tilde{v}$  of any holomorphic vector field  $v$  on  $X$ . For every  $g_t = e^{tv}$  in the one parameter subgroup generated by  $v$ , we get that  $\tilde{g}_t := e^{t\tilde{v}}$  commutes with every covering transformation  $h$ . Therefore

$$\tilde{g}_t h_* \tilde{\sigma} = h_* \tilde{g}_t \tilde{\sigma} = \rho(h)\tilde{\sigma}.$$

The meromorphic function  $F = \frac{\tilde{\sigma}}{\tilde{g}_t \tilde{\sigma}}$  descends to  $X$  as a meromorphic function  $f$  such that  $(f)_0 = \Delta$  and  $(f)_{\infty} = g_t(\Delta)$ .  $\square$

The  $\mathbb{C}^s$ -action induced by the splitting  $TX = S \oplus S^{\perp}$  is determined by  $s$  commuting holomorphic vector fields  $v_1, \dots, v_s$  on  $X$  linearly independent everywhere. Let us call  $G \in \text{Aut}_0(X)$  the (non necessarily closed) subgroup determined by the abelian Lie algebra spanned by  $v_1, \dots, v_s$  and  $\overline{G}$  be its Zariski closure in  $\text{Aut}_0(X)$ .

According to [37], there is an exact sequence of groups

$$1 \rightarrow H \rightarrow \overline{G} \rightarrow \text{Aut}_0(T)(=T) \rightarrow 1$$

where  $H$  is a linear algebraic group with Lie algebra  $\mathfrak{h}$  contained in the ideal of holomorphic vectors field tangent to the rationally connected fibration.

**Lemma 8.5.** *Up to replacing  $v_1, \dots, v_s$  by  $v_1 + \xi_1, \dots, v_s + \xi_s$  for suitable  $\xi_1, \dots, \xi_s \in \mathfrak{h}$ , one may assume that the natural action of  $\overline{G}$  on  $\mathbb{P}(|\mathcal{O}(\Delta)|)$  is trivial.*

*Proof.* Let  $\rho$  be the natural group morphism  $\overline{G} \rightarrow \text{Aut } \mathbb{P}(|\mathcal{O}(\Delta)|)$ . Remark that  $\rho$  is well defined by virtue of lemma 8.4. We call  $\rho_*$  the induced morphism at the Lie algebra level.

Since  $T$  is a torus, every morphism  $T \rightarrow \rho(\tilde{G})/\rho(H)$  is trivial. It follows that for every  $1 \leq i \leq s$ , there exists  $\xi_i \in \mathfrak{h}$  such that  $\rho_*(v_i) + \rho_*(\xi_i) = 0$ . The lemma follows.  $\square$

Taking again a finite etale covering if necessary, one can assume that  $N\mathcal{F}$  is linearly equivalent to  $\Delta$ , so the foliation is defined by a rational closed one form  $\omega$  such that  $(\omega)_0 = (0)$  and  $(\omega)_\infty = \Delta$ . With the same arguments as previously, one can prove the following lemma.

**Lemma 8.6.** *One can moreover choose  $\xi_1, \dots, \xi_s$  in the previous lemma such that  $\overline{G}$  acts trivially on  $H^0(X, \Omega^1 \otimes \mathcal{O}(\Delta))$  (that is, the space of one rational form  $\eta$  with  $(\eta)_\infty \leq \Delta$ ).*

**8.3. Proof of Theorem 8.1.** Using the previous lemmata, we can assert that  $\omega$  is invariant under the action of  $\overline{G}$ . Since  $\omega$  is closed, for every  $v \in \mathfrak{g} := \text{Lie}(\overline{G})$  we have  $\omega(v)$  constant. Taking suitable linear combinations, one can assume that  $\omega(v_i) = 0$  for  $i = 1, \dots, s-1$ . There are two possibilities:

- (1)  $\mathfrak{h}$  is non trivial, so there exists  $v \in \mathfrak{h}$  such that  $\omega(v) = 1$  (otherwise,  $\mathcal{F}$  would be uniruled) and we obtain (2), replacing  $v_s$  by  $v_s - \omega(v_s)v$ ;
- (2)  $\mathfrak{h} = \{0\}$  and in this case we obtain (3).

Theorem 8.1 follows.  $\square$

**8.4. The canonical bundle of  $\mathcal{F}$  is torsion.** It is natural to enquire if the flatness of  $K\mathcal{F}$  implies that it is torsion. We are not aware of any example where this is not the case, and we can prove this is the case under the assumptions of Theorem 8.1.

**Corollary 8.7.** *Assume that the foliation  $\mathcal{F}$  fulfills the hypothesis of Theorem 8.1. Then  $KX \simeq \mathcal{O}(-\Delta)$  modulo torsion, so  $K\mathcal{F}$  is actually a torsion line bundle.*

*Proof.* First, observe that we can restrict to the case  $X = Y$  in the notation used in §8.1. The universal covering  $\pi : \tilde{Y} \rightarrow Y$  has the form  $\tilde{Y} = \mathbb{C}^s \times F$  and covering transformations act as  $h(x, y) = (h_1(x), h_2(y))$ , where  $h_1$  is a translation and  $h_2 \in \text{Aut}(F)$ . We can also assume that  $h_2$  lies in the connected component of the identity (see [6]).

Noticing that  $\Delta$  is  $\overline{G}$  invariant (after a suitable modification described in the previous lemmata), we can conclude that there exists on  $F$  an effective divisor  $\Delta'$  such that  $\pi^{-1}(\Delta) = \Delta' \times \mathbb{C}^s$ ,  $KF \simeq \mathcal{O}(-\Delta')$ .

Let  $\eta$  a rational meromorphic volume form, expressed as a non trivial section of  $KF \otimes \mathcal{O}(\Delta')$ . Therefore

$$h_2^* \eta = \lambda_{h_2} \eta,$$

where  $\lambda_{h_2} \in \mathbb{C}$ . If we take the Zariski closure in  $\text{Aut}(F)$  of the group generated by the second component of the deck transformations then, in case  $K\mathcal{F}$  is not torsion, we produce a holomorphic vector field in  $F$  such that  $L_v \eta = \lambda \eta$  for a suitable  $\lambda \in \mathbb{C}^*$ . Since  $\eta$  has at least one pole of order 1, the corollary will be a consequence of the lemma below.  $\square$

**Lemma 8.8.** *Let  $M$  be a complex compact manifold of dimension  $n$ ,  $v$  an holomorphic vector field and  $\eta$  a meromorphic section of the canonical bundle  $KM$  with at*

least a pole of order one. Assume moreover there exists  $\lambda \in \mathbb{C}$  such that  $L_v \eta = \lambda \eta$ . Then,  $\lambda = 0$  (equivalently,  $(e^{tv})^* \eta = \eta$  for every  $t \in \mathbb{C}$ ).

*Proof.* Let be  $D$  a simple pole of  $\eta$  and assume for a moment that  $D$  and only intersects with normal crossings  $(\eta)_\infty$  along simple poles. Then, there is a well defined non zero residue  $\alpha$ , in this context a  $n - 1$  holomorphic form on  $D$  (see for instance [2]). Let  $\lambda_t$  such that  $(e^{tv})^* \eta = \lambda_t \eta$ ; we get also

$$(e^{tv})^* \alpha = \lambda_t \alpha$$

with the first term of the equality above making sense because  $v$  is necessarily tangent to  $D$ . Using now that  $\int_D \alpha \wedge \bar{\alpha}$  is invariant under small biholomorphisms of  $D$ , we obtain that  $|\lambda_t| = 1$ , for all  $t \in \mathbb{C}$ , hence  $\lambda_t = 1$ .

The general case can be reduced to the one treated above by taking a suitable composition of blowing ups with smooth centers contained in the singular set of  $(\eta)_\infty$ .  $\square$

#### REFERENCES

1. J. AMOROS, M. MANJARIN, M. NICOLAU, *Deformations of Kähler manifolds with non vanishing holomorphic vector fields*. arXiv:0909.4690v4 [math.AG] (2010) to appear in JEMS. 4.2
2. W. BARTH, C. PETERS, A VAN DE VEN, *Compact complex surfaces*. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 4. Springer-Verlag, Berlin, 1984. 8.4
3. P. BAUM, R. BOTT, *Singularities of holomorphic foliations*. J. Differential Geometry **7** (1972), 279–342. 3.4
4. A. BEAUVILLE, *Variétés Kähleriennes dont la première classe de Chern est nulle*. J. Differential Geom. **18** (1983), no. 4, 755–782 (1984). 8.1
5. A. BEAUVILLE, *Holomorphic symplectic geometry: a problem list*. Preprint arXiv:1002.4321. To appear in the Proceedings of the conference “Complex and Differential Geometry” (Hanovre, 2009). 4.8
6. A. BLANCHARD, *Sur les variétés analytiques complexes*. Ann. Sci. Ecole Norm. Sup. (3) **73** (1956), 157–202. 8.2, 8.4
7. F. BOGOMOLOV, *The decomposition of Kähler manifolds with a trivial canonical class*. Mathematics of the USSR-Sbornik **22** (4) (1974), 580–583. 4.1
8. F. BOGOMOLOV, M. MCQUILLAN, *Rational Curves on Foliated Varieties*. Preprint IHES M/01/07 (2001). 1.5, 2.3, 6.7
9. P. BONNET, *Minimal invariant varieties and first integrals for algebraic foliations*. Bull. Braz. Math. Soc. (N.S.) **37** (2006), 1–17. 6.4
10. J.-B. BOST, *Algebraic leaves of algebraic foliations over number fields*. Publ. Math. Inst. Hautes Études Sci. No. **93** (2001), 161–221. 1.3, 3.1
11. S. BOUCKSOM, J.-P. DEMAILLY, M. PAUN, T. PETERNELL, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*. arXiv:math/0405285v1 [math.AG] (2004). 1.4, 3.4, 4.2, 7.3.1
12. M. BRUNELLA, *Birational Geometry of Foliations*. Publicações Matemáticas do IMPA. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2004. 1, 3.4, 5.1, 5.1
13. M. BRUNELLA, *A positivity property for foliations on compact Kähler manifolds*. Internat. J. Math. **17** (2006), no. 1, 35–43. 4.3
14. M. BRUNELLA, J.V. PEREIRA, F. TOUZET, *Kähler manifolds with split tangent bundle*. Bull. Soc. Math. France **134** (2006), no. 2, 241–252. 4.1
15. F. CAMPANA, T. PETERNELL, *Geometric stability of the cotangent bundle and the universal cover of a projective manifold (with an appendix by Matei Toma)*. Bull. Soc. math. France **139** (2011) p. 41–74. 3.4, 4.2
16. G. CASALE, *Suites de Godbillon-Vey et intégrales premières*. C. R. Math. Acad. Sci. Paris **335** (2002) p.1003–1006. 5.2
17. D. CERVEAU, A. LINS NETO, *Irreducible components of the space of holomorphic foliations of degree two in  $\mathbb{CP}(n)$ ,  $n \geq 3$* . Ann. of Math. **143** (1996) p.577–612. 1

18. D. CERVEAU, A. LINS NETO, F. LORAY, J. V. PEREIRA, F. TOUZET, *Algebraic Reduction Theorem for complex codimension one singular foliations*. Comment. Math. Helv. **81** (2006) p.157-169. 1.5, 5.2
19. D. CERVEAU, A. LINS NETO, F. LORAY, J. V. PEREIRA, F. TOUZET, *Complex Codimension one singular foliations and Godbillon-Vey Sequences*. Moscow Math. Jour. **7** (2007) p.21-54. 1.2, 1.3, 1.5, 1.7, 3.2, 3.2, 5.2, 6.10
20. D. CERVEAU, A. LINS NETO, *A structural theorem for codimension one foliations on  $\mathbb{P}^n$ ,  $n \geq 3$ , with application to degree three foliations*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (to appear). 1.5, 3.4, 3.4
21. S. C. COUTINHO, J. V. PEREIRA, *On the density of algebraic foliations without algebraic invariant sets*. J. Reine Angew. Math. **594** (2006), 117–135. 3.1
22. O. DEBARRE, *Higher-dimensional algebraic geometry*. Universitext. Springer-Verlag, New York, 2001. xiv+233 pp. 6.1
23. J.-P. DEMAILLY, T. PETERNELL, M. SCHNEIDER, *Compact Kähler manifolds with Hermitian semipositive anticanonical bundle*. Compositio Math. **101** (1996), no. 2, 217–224. 8, 8.1
24. J.-P. DEMAILLY, *On the Frobenius integrability of certain holomorphic  $p$ -forms*. Complex geometry (Göttingen, 2000), 93–98, Springer, Berlin, 2002. 4.1, 4.1
25. A. DE MEDEIROS, *Singular foliations and differential  $p$ -forms*. Ann. Fac. Sci. Toulouse Math. (6) **9** (2000), no. 3, 451–466. 3.4
26. S. DRUEL, *Structures de Poisson sur les variétés algébriques de dimension 3*. Bull. Soc. Math. France **127** (1999), no. 2, 229–253. 1.1
27. T. EKEDEHL, N.I. SHEPHERD-BARON, R.L TAYLOR, *A conjecture on the existence of compact leaves of algebraic foliations*, Shepherd-Baron’s homepage. 1.3, 3.1, 3.1
28. T. GRABER, J. HARRIS, J. STARR, *Families of rationally connected varieties*. Journal of AMS **16** (2003), no.5, 57–67. 7.3.1
29. H. GRAUERT and R. REMMERT, *Theory of Stein spaces*. Springer-Verlag (1979). 4.3
30. R. HARTSHORNE, *Ample vector bundles on curves*. Nagoya Math. J. **43** (1971), 73–89. 2.3
31. R. HARTSHORNE, *Cohomological dimension of algebraic varieties*. Ann. of Math. (2) **88** (1968) 403–450. 1.5, 6.8
32. N. JACOBSON, *Lie algebras*. Republication of the 1962 original. Dover Publications, Inc., New York, 1979. 3.2, 3.3
33. S. KEBEKUS, L. SOLÁ CONDE, M. TOMA, *Rationally connected foliations after Bogomolov and McQuillan*. J. Algebraic Geom. **16** (2007), no. 1, 65–81. 2.3
34. J.KOLLÁR *et al*, *Flips and abundance for algebraic threefolds*, Astérisque **211** (1993) 2.2, 2.3, 2.4, 2.4, 3.3
35. J. KOLLÁR, *Rational curves on algebraic varieties*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, **32**. Springer-Verlag, Berlin, 1996. viii+320 pp. 6.1, 6.6, 6.7
36. J. KOLLÁR, *Fundamental groups of rationally connected varieties*. Michigan Math. J. **48** (2000), 359–368. 1.5, 6.9, 6.10
37. D. LIEBERMAN *Compactness of the chow scheme: applications to automorphisms and deformations of Kähler manifolds*. Fonctions de plusieurs variables complexes, III (Sém. F. Norguet, 1975-77), pp. 140-186, Lecture Notes in Math, **670**. 8.2
38. F. LORAY, *A preparation theorem for codimension-one foliations*. Ann. of Math. (2) **163** (2006), no. 2, 709–722. 3.4
39. F. LORAY AND J.V. PEREIRA, *Transversely projective foliations on surfaces: existence of normal forms and prescription of the monodromy*. Intern. Jour. Math. **18** (2007) p.723-747. 5.1, 5.2, 6.6
40. F. LORAY AND D. MARÍN PÉREZ, *Projective structures and projective bundles over compact Riemann surfaces*. Astérisque No. **323** (2009), 223-252. 5.1
41. F. LORAY, J. V. PEREIRA AND F. TOUZET, *Foliations with trivial canonical bundle on Fano 3-folds*. Preprint, 2011. 3.4
42. J. MARTINET, *Normalisation des champs de vecteurs holomorphes (d’après A.-D. Brjuno)*. Bourbaki Seminar, Vol. 1980/81, pp. 55–70, Lecture Notes in Math., **901**, Springer, Berlin-New York, 1981 3.4, 3.4
43. M. MCQUILLAN, *Canonical models of foliations*. Pure Appl. Math. Q. **4** (2008), no. 3, part 2, 877–1012. 1, 3.4, 3.4

44. V. B. MEHTA, A. RAMANATHAN, *Semistable sheaves on projective varieties and their restriction to curves*. Math. Ann. **258** (1981/82), no. 3, 213–224. 2.3, 4.2
45. Y. MIYAOKA, *Deformations of a morphism along a foliation and applications*. Algebraic geometry, Bowdoin, 1985, Proc. Sympos. Pure Math., **46**, Part 1, Amer. Math. Soc., Providence, RI (1987) p. 245–268. 2.3, 6.3
46. Y. MIYAOKA AND T. PETERNELL, *Geometry of higher dimensional algebraic varieties*. DMV Seminar, **26**. Birkhäuser Verlag, Basel, 1997. vi+217 pp. 3.1, 3.1, 6.7
47. S. NEUMANN, *A decomposition of the Moving cone of a projective manifold according to the Harder-Narasimhan filtration of the tangent bundle*. Phd thesis. Universitat Freiburg, (2009). Available at [http://www.freidok.uni-freiburg.de/volltexte/7287/pdf/Diss\\_Neumann.pdf](http://www.freidok.uni-freiburg.de/volltexte/7287/pdf/Diss_Neumann.pdf) 3.4
48. T. PETERNELL, *Minimal varieties with trivial canonical classes. I*. Math. Z. **217** (1994), no. 3, 377–405. 4.1
49. T. PETERNELL, *Generically nef vector bundles and geometric applications*. arXiv:1106.4241v1 [math.AG]. 1, 1.4, 4.1, 4.2
50. A. POLISHCHUK, *Algebraic geometry of Poisson brackets*. Algebraic geometry, 7. J. Math. Sci. **84** (1997), no. 5, 1413–1444. 1, 1.1
51. H. P. DE SAINT-GERVAIS, *Uniformisation des surfaces de Riemann. Retour sur un théorème centenaire*. ENS Éditions, Lyon, 2010. 5.1
52. B. SCARDUA, *Transversely affine and transversely projective holomorphic foliations*, Ann. Sci. École Norm. Sup. (4) **30** (1997), no. 2, 169–204. 5.2
53. N. I. SHEPHERD-BARRON, *Semi-stability and reduction mod  $p$* . Topology **37** (1998), no. 3, 659–664. 3.3
54. C. SIMPSON, *Subspaces of moduli spaces of rank one local systems*. Ann. Sci. École Norm. Sup. (4) **26** (1993), no. 3, 361–401. 1.4, 4.1
55. A. J. SOMMESE, *Holomorphic vector-fields on compact Kähler manifolds*, Math. Ann. **210** (1974), 75–82. 4.3
56. F. TOUZET, *Feuilletages holomorphes de codimension un dont la classe canonique est triviale*. Ann. Sci. Éc. Norm. Supér. (4) **41** (2008), no. 4, 655–668. 1, 1.1, 4.1, 7.3
57. F. TOUZET, *Structure des Feuilletages Kählériens en courbure semi-négative*. Annales de la faculté des sciences de Toulouse **19**, numéro 3-4 (2010), 865–886. 4.1
58. F. TOUZET, *Uniformisation de l'espace des feuilles de certains feuilletages de codimension un*. Available at arXiv:1103.4626v1 [math.CV]. 4.1
59. F. TOUZET, *Sur les feuilletages holomorphes transversalement projectifs*. Ann. Inst. Fourier (Grenoble) **53** (2003), no. 3, 815–846. 6.10
60. S.-T. YAU, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*, Comm. Pure Appl. Math. **31** (1978) 339–411. 8.1

1 IRMAR, CAMPUS DE BEAULIEU, 35042 RENNES CEDEX, FRANCE

2 IMPA, ESTRADA DONA CASTORINA, 110, HORTO, RIO DE JANEIRO, BRASIL

E-mail address: <sup>1</sup> [frank.loray@univ-rennes1.fr](mailto:frank.loray@univ-rennes1.fr), [frederic.touzet@univ-rennes1.fr](mailto:frederic.touzet@univ-rennes1.fr)

E-mail address: <sup>2</sup> [jvp@impa.br](mailto:jvp@impa.br)